3SUM in Preprocessed Universes

FASTER AND SIMPLER

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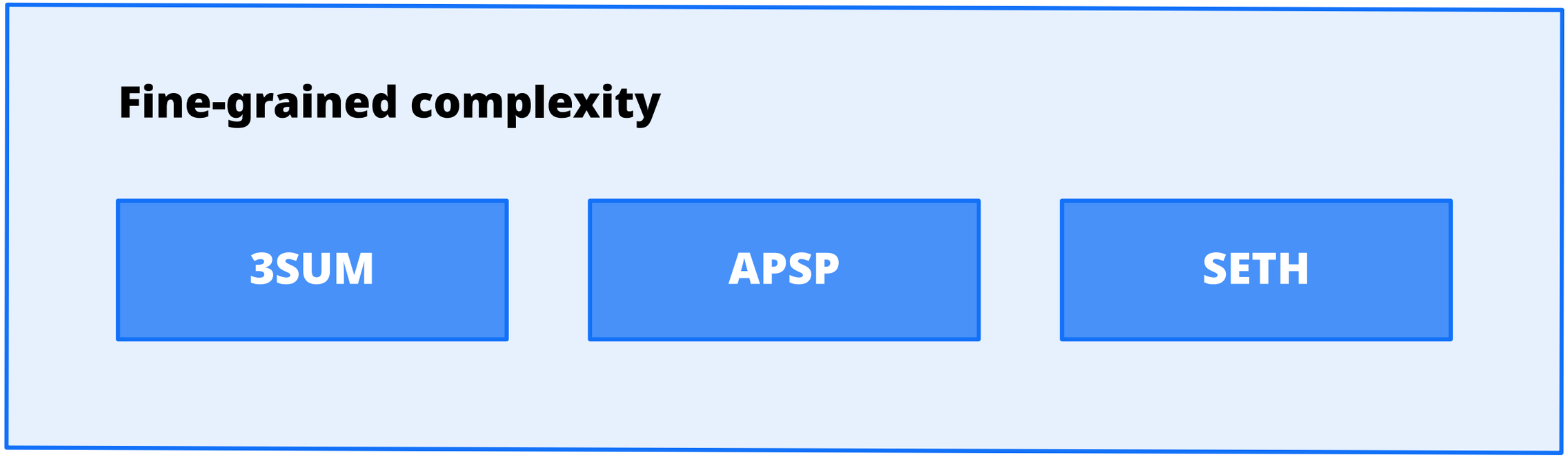
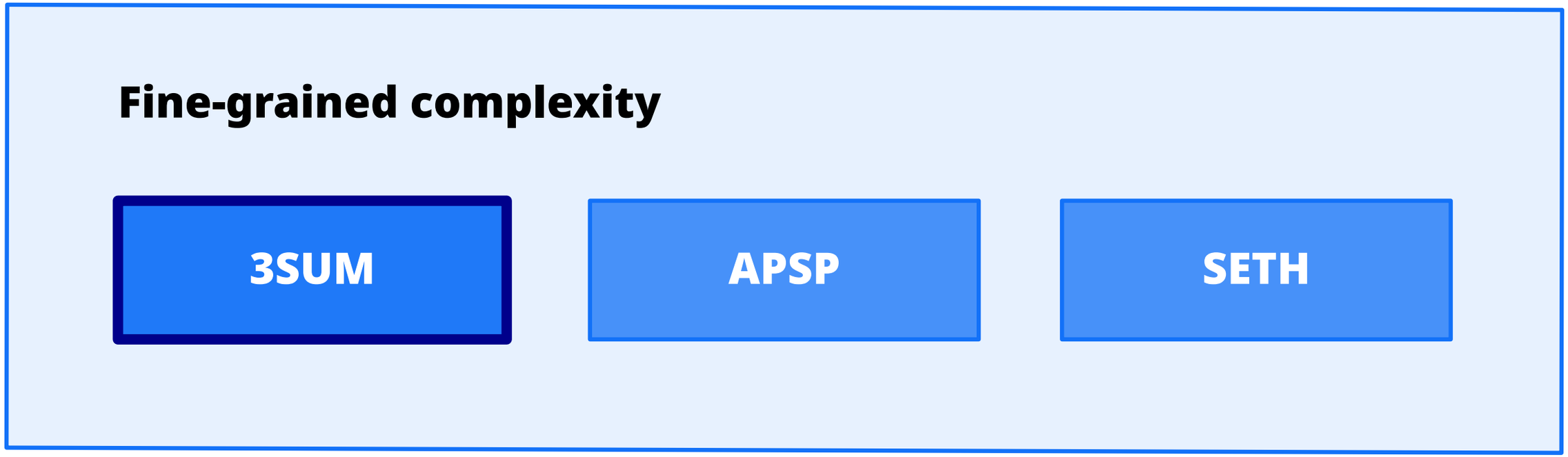

Fine-grained complexity
3SUM
APSP
SETH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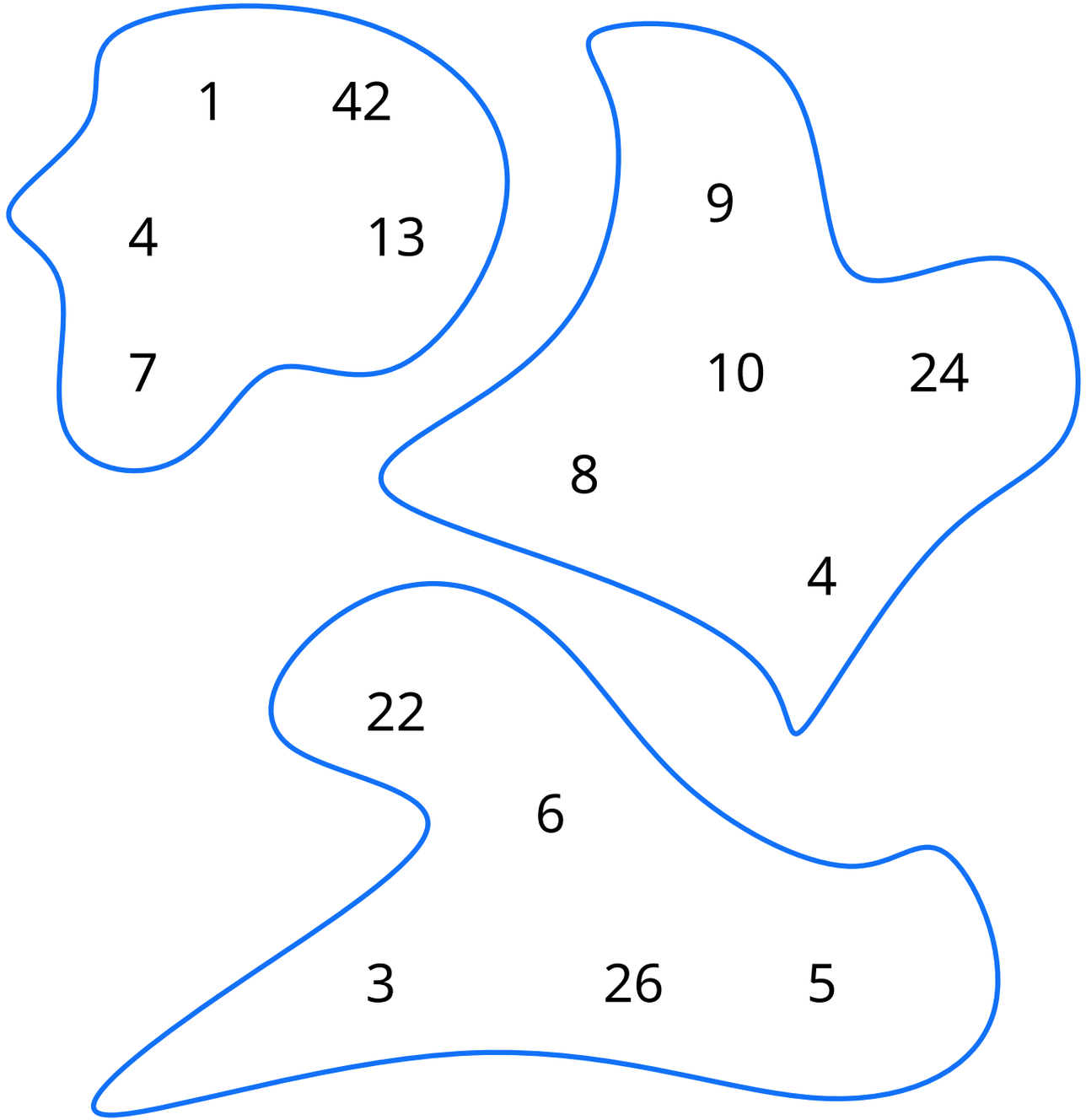
The 3SUM problem

Input:
Three sets A, B, C of n integers

Output:
$\exists\, a \in A, b \in B, c \in C$ with $a + b = c$ **?**



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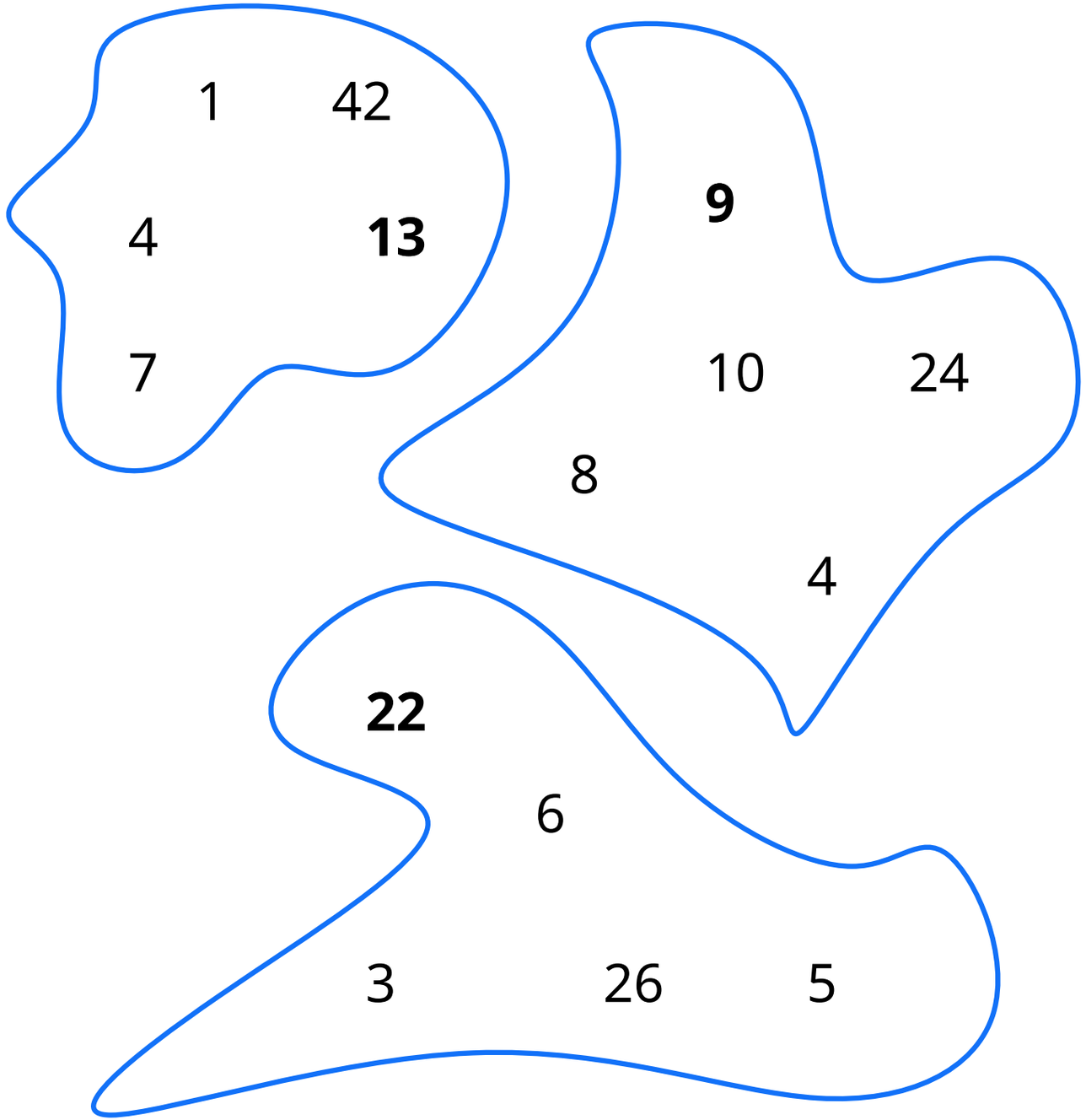

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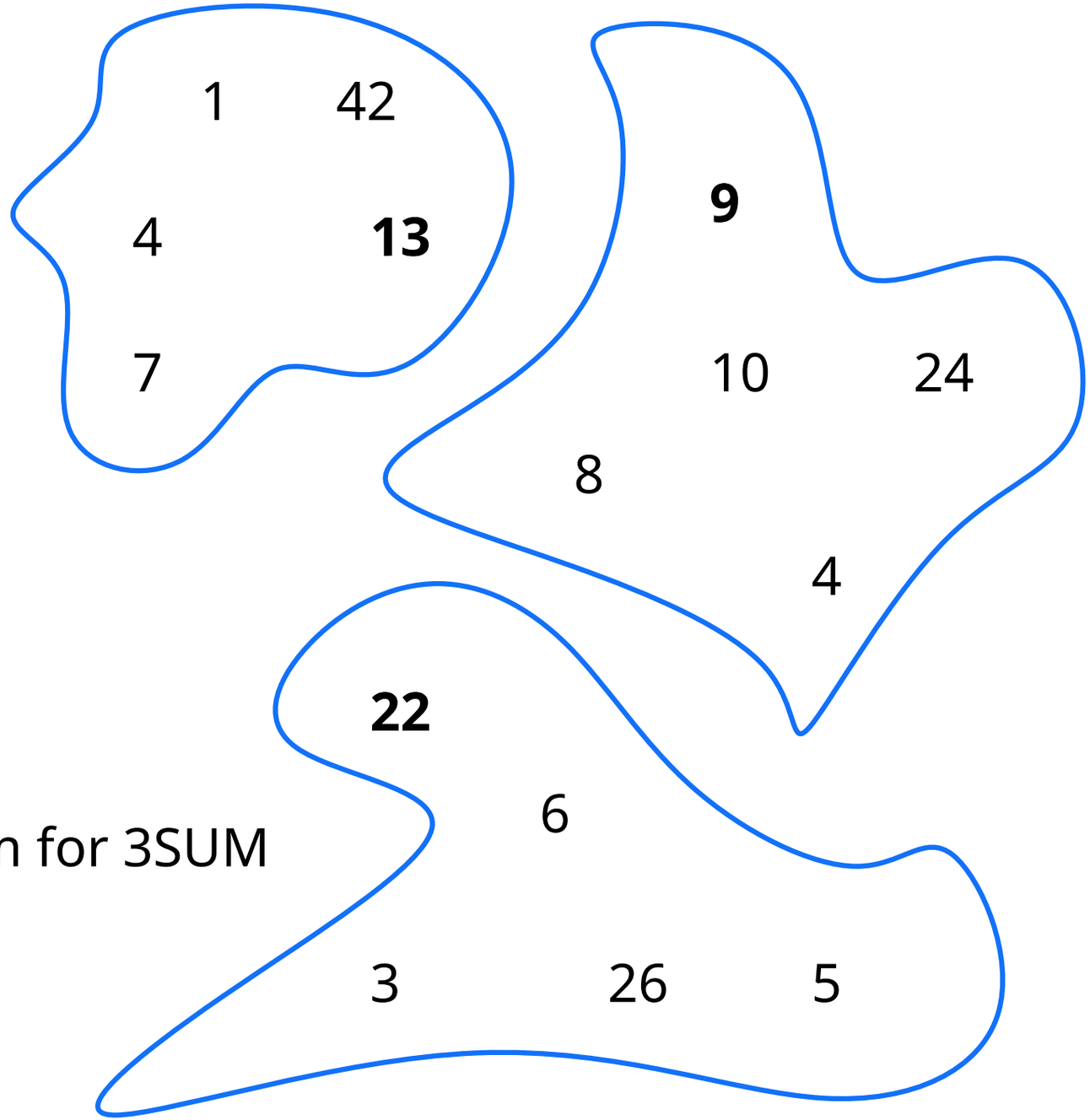

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Hypothesis: No $O(n^{1.99})$ time algorithm for 3SUM

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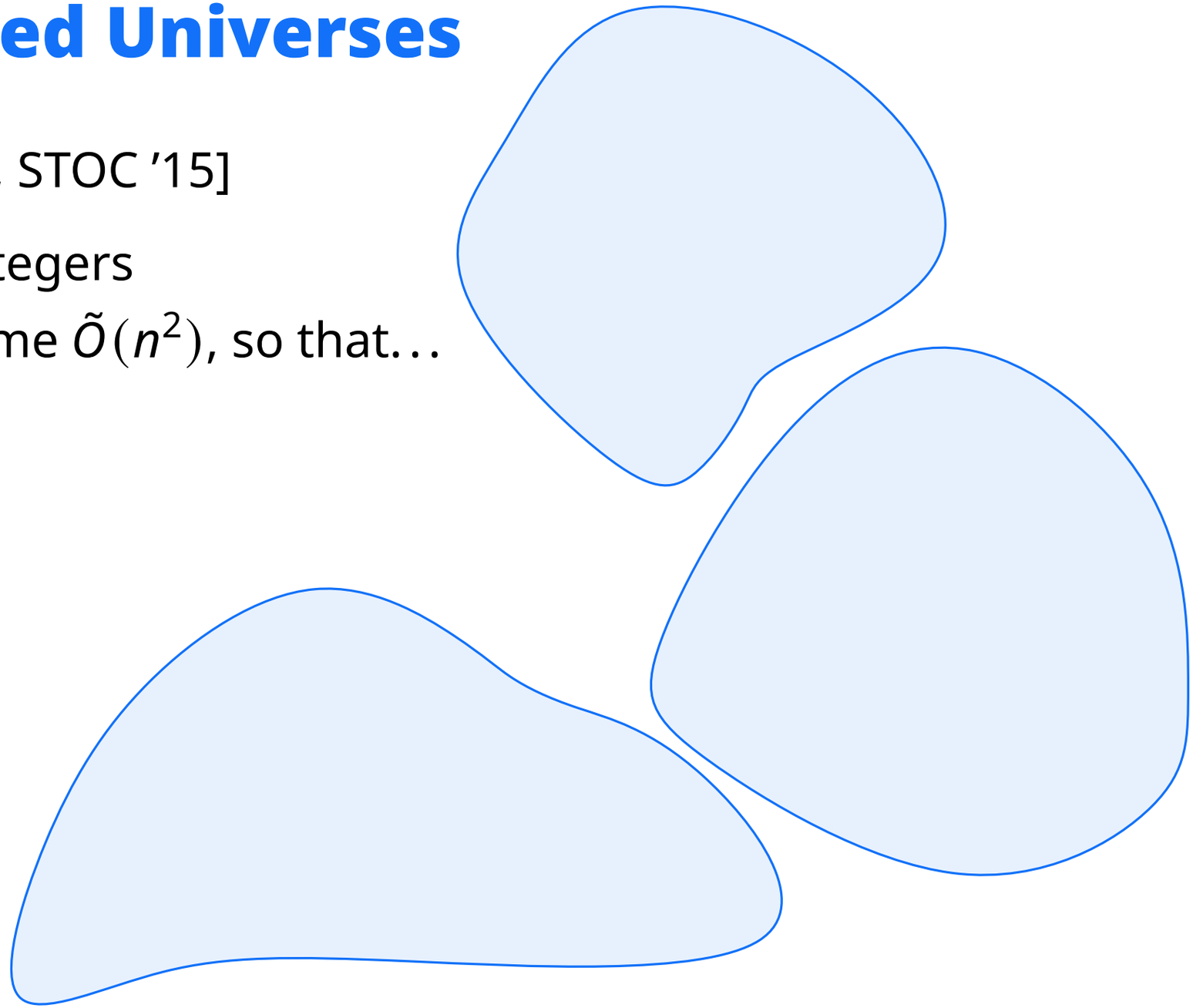
Theorem [Chan–Lewenstein, STOC '15]

Given three sets A, B, C of n integers

one can preprocess them, in time $\tilde{O}(n^2)$, so that…



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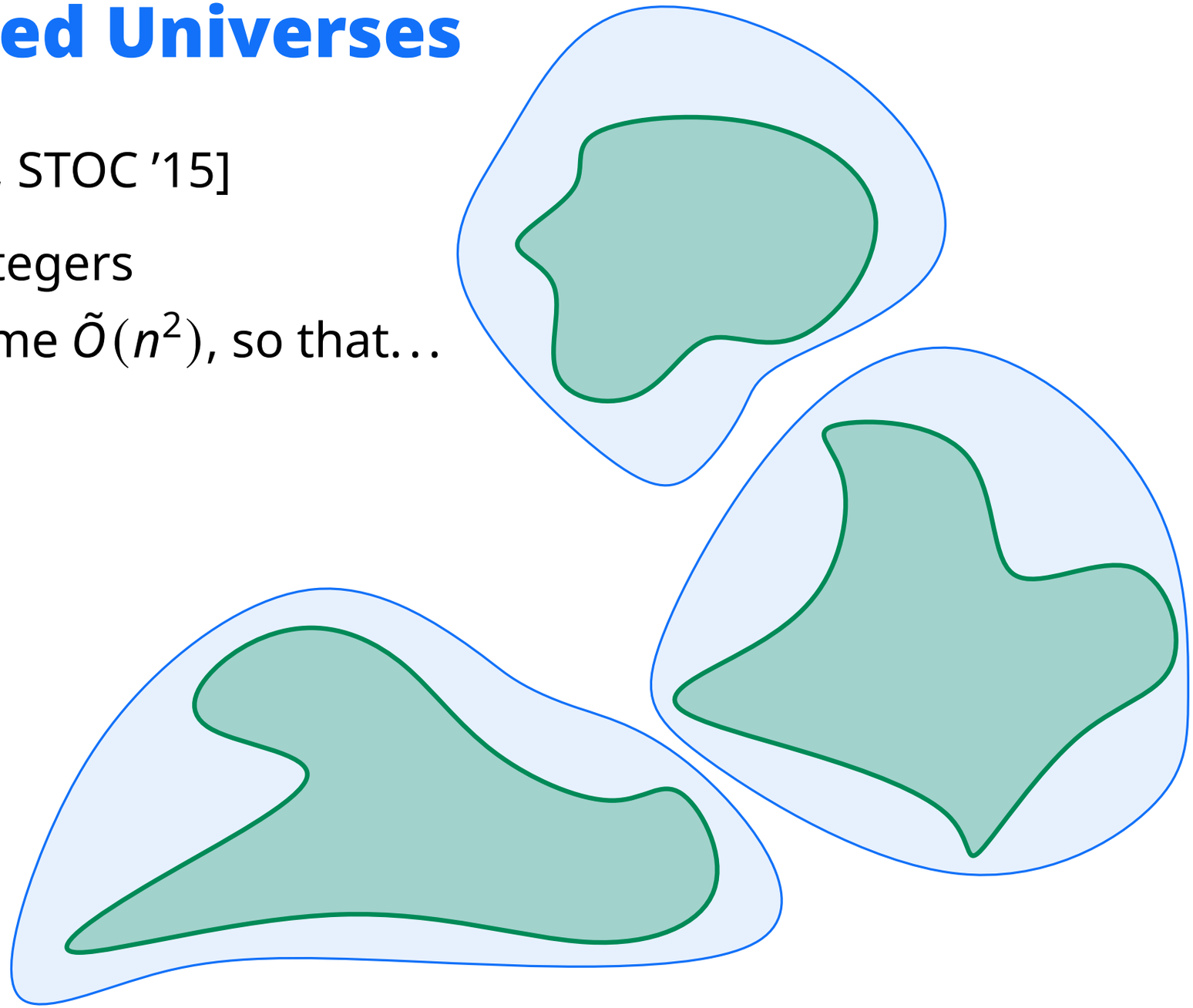

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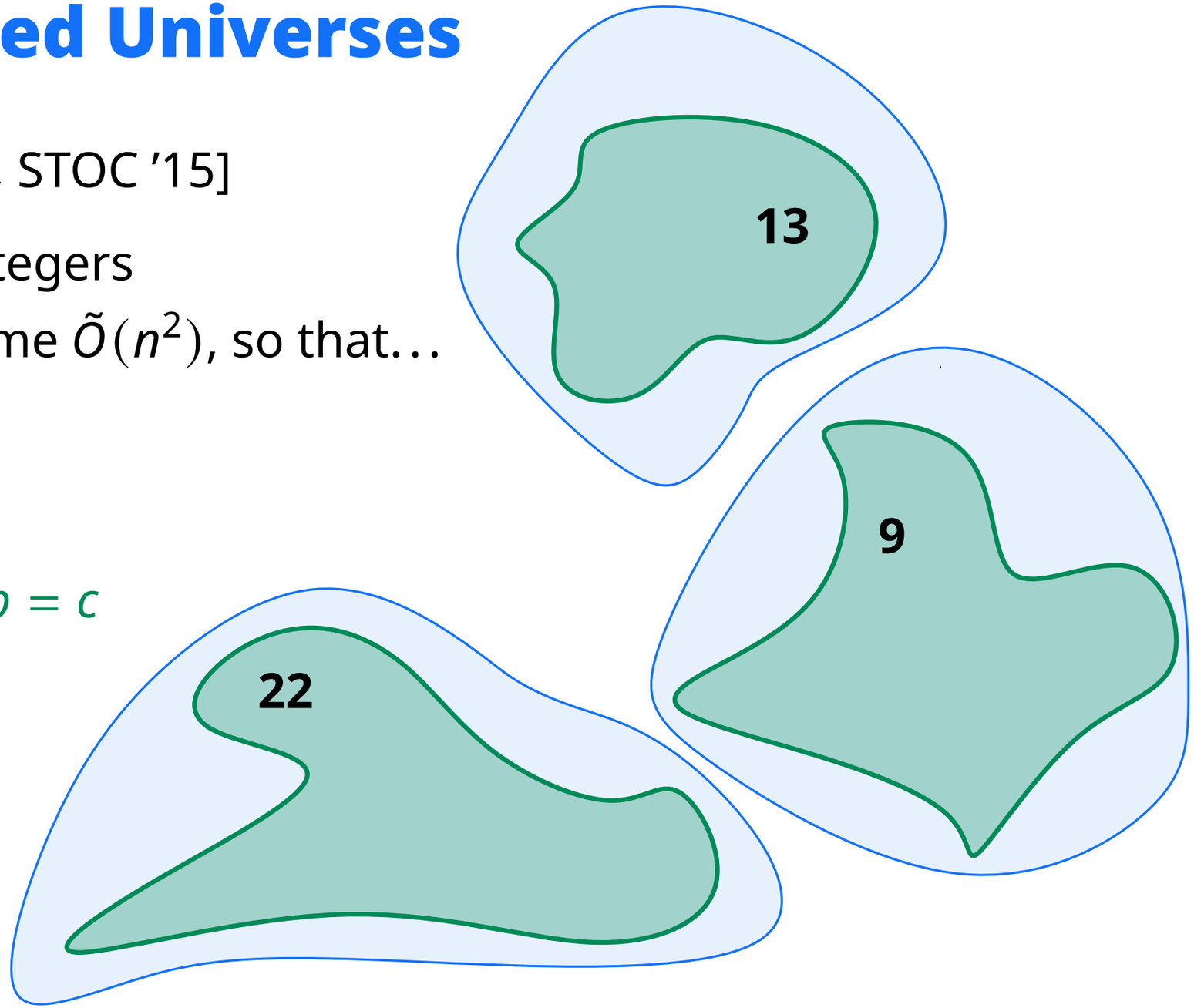
one can decide if there exists

$a \in A'$, $b \in B'$, $c \in C'$ with $a + b = c$

in time $\tilde{O}(n^{13/7})$.



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elementary

Tool: linear hashing modulo p

$$\{(a, b, c) \in A \times B \times C : a + b \equiv c \bmod p\}$$

solutions modulo p

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solutions modulo p $=$ true solutions $\cup$ false positives

$$\{(a,b,c) \in A \times B \times C : a + b = c\}$$

$$\{(a,b,c) \in A \times B \times C : a + b \equiv c \bmod p \wedge a + b \neq c\}$$

Our preprocessing

1. Pick a random prime $p \in [n^{1.5}, 2 \cdot n^{1.5})$

2. List all **false positives** **modulo** p

$$F = \{(a, b, c) \in A \times B \times C : a + b \equiv c \bmod p \wedge a + b \neq c\}$$

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in time $O(n^2 + |F|)$

Observation: $\mathbb{E}[|F|] \leqslant O(n^{1.5} \log n)$

Bounding the number of false positives

$$F = \{(a, b, c) \in A \times B \times C : a + b \equiv c \bmod p \wedge a + b \neq c\}$$

$$\mathbb{E}[|F|] = \sum_{a+b \neq c} \Pr[a + b \equiv c \bmod p]$$

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$$= O(n^{1.5} \log n) \quad \blacksquare$$

Our algorithm

1. Calculate **number** of **solutions modulo** p, i.e.,

$$\#\left\{(a, b, c) \in A' \times B' \times C' : a + b \equiv c \bmod p\right\}$$

2. Subtract **false positives** present in the **actual instance**, i.e.,

$$F \cap (A' \times B' \times C')$$

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in time $O(p \log p)$, using **FFT**

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$$Q_A(x) = \sum_{a \in A} x^a$$

e.g.: $Q_{\{2,3,7\}} = x^2 + x^3 + x^7$

$$(Q_A \cdot Q_B)(x) = \sum \alpha_c x^c$$

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Note: Can output for **each** $c \in C'$ if there exist $a \in A'$, $b \in B'$ with $a + b = c$

Derandomizing our preprocessing

Need: A **modulus** $m \approx n^{1.5}$ s.t. **# false positives modulo** m is $|F_m| \approx n^3/m$

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$$\widetilde{O}(n^{1.5} \cdot n^{1.5}) = \widetilde{O}(n^3) \quad :-(

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[Chan–He '20, Fischer–Kaliciak–P. '24]

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    - Is even our algorithm "combinatorial"? [Pratt–Uffenheimer–Weinstein]

THANK YOU!