BIN PACKING WITH FEW SMALL ITEMS
VIA FAST EXACT MATCHING IN MULTIGRAPHS

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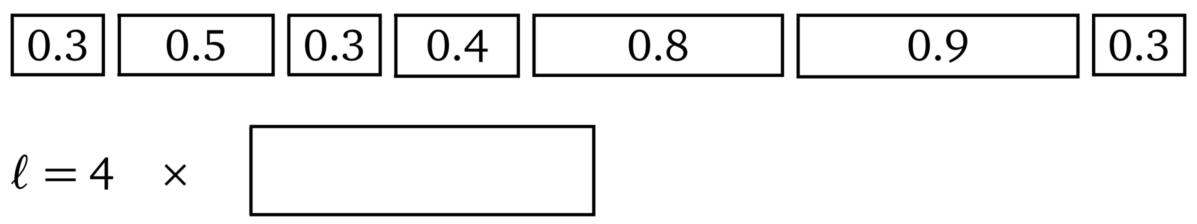
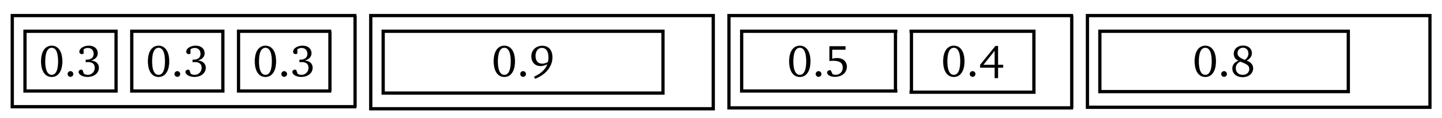
The Bin Packing problem

Given n items $x_1, x_2, \ldots, x_n \in \mathbb{Q}_{\geqslant 0}$, and number ℓ of unit-sized bins,



decide if the items can be packed into the bins.

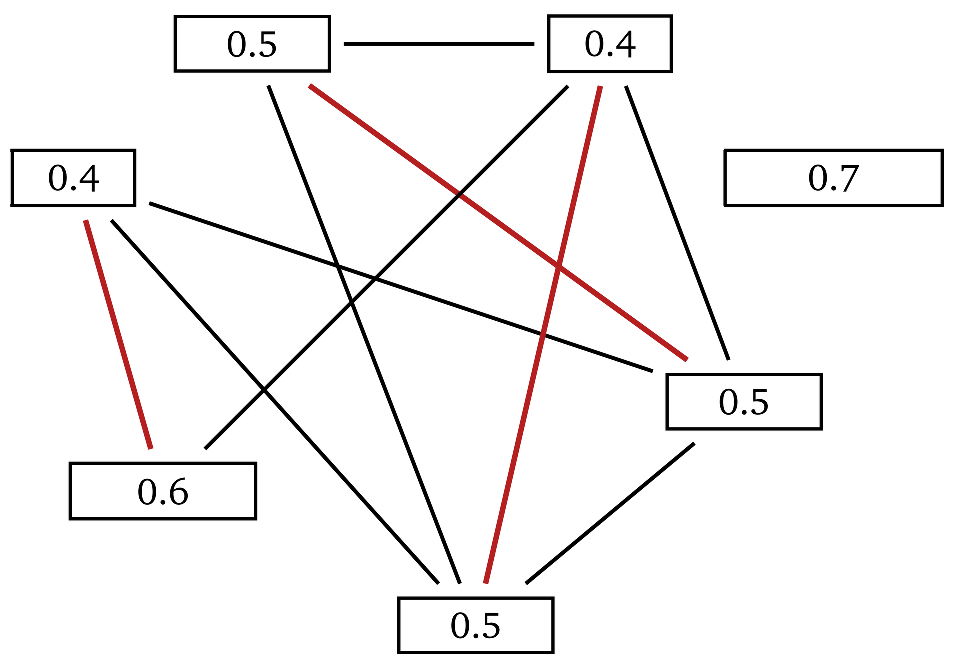


Bin packing is **NP-hard**, but…

…Bin Packing is easy when all items are large

$\forall_i\ x_i > 1/3 \implies$ at most **two** items per bin

Matching in compatibility graph on items: edge $x \leftrightarrow j$ iff $x_i + x_j \leqslant 1$



$\#\text{bins} = n - \#\text{matched edges}$

Bin Packing with Few Small Items

Bin Packing parameterized by the number of small items $k = \underbrace{\#\{i \mid x_i \leqslant 1/3\}}_{\text{"distance to triviality"}}$

Distance to triviality is a popular parameter in FPT literature, e.g.:

- Vertex Cover parameterized above matching
- Dominating Set parameterized by treewidth
- Longest Common Subsequence parameterized by maximum occurrence number

Our main result

Bin Packing with Few Small Items can be solved in $O^*(2^k)$ time.

Two ingredients:

- Reduction to Exact Weighted Perfect Matching
- Careful variant of Mulmuley–Vazirani–Vazirani exact matching algorithm

Previous best: $\boldsymbol{O^*(k!4^k)}$ [Bannach et al., 2020]

What else is in the paper?

- $O^*(2^k)$ time algorithms for
  - **Vector**† Bin Packing with Few Small Items
  - (Vector) Bin **Covering** with Few Small Items
  - (Vector) **Multiple Knapsack** with Few Small Items
  - Perfect Matching with **Hitting Constraints**
- Lower bound under SETH
  - no $O^*(2^{(1-\varepsilon)n})$ time algorithm for Vector Bin Packing‡
  - no $O^*(2^{(1-\varepsilon)k})$ time algorithm for Vector Bin Packing with Few Small Items

† items from $\mathbb{Q}^d_{\geqslant 0}$ require a more complex notion of large items [Bannach et al., 2020]

‡ for (one-dimensional) Bin Packing no c^n lower bound is known (similarly to Subset Sum), and it is an important open problem to find one

This is not a Pfaffian.

Bin Packing with Few Small Items → Exact Matching

1. Add $2\ell - (n-k)$ dummy zero-sized "large" items
   $\implies$ Exactly two large items in each bin
2. Create compatibility (multi-)graph on large (and dummy) items

for $i \in$ large items **do**
  for $j \in$ large items **do**
    for $S \in 2^{\text{small items}}$ **do**
      if $x_i + x_j + \sum_{s \in S} x_s \leqslant 1$ **then**
        add edge $i \leftrightarrow j$ with weight $f(S) = \underbrace{|S| \cdot 2^k}_{\text{bitcount}} + \underbrace{\sum_{i \in S} 2^i}_{\text{bitmask}}$

$O(n)$ nodes
$O(2^k \cdot n^2)$ edges
$O(2^k \cdot k)$ max weight

3. Find perfect matching of total weight $k2^k + (2^k - 1)$

Lemma: $f(S_1) + f(S_2) + \cdots + f(S_m) = k2^k + (2^k - 1)$
$\iff S_1, S_2, \ldots, S_m$ is a partition of $[k]$

Exact Matching in multigraphs with large weights

Vanilla Mulmuley–Vazirani–Vazirani time: poly(#nodes, #edges, max weight)

Our target running time: poly(#nodes) · (#edges + max weight)

Idea: apply Isolation Lemma to $\underbrace{\text{pairs of nodes}}_{n^2}$ instead of $\underbrace{\text{(multi-)edges}}_{2^k n^2}$.

Algorithm: compute $\underset{\substack{\uparrow \\ \text{like determinant, but cooler!}}}{\text{Pfaffian}}$ pf(A) of adjacency matrix.

$$A = \begin{bmatrix} & & \\ & & A_{ij} = \underset{\substack{\uparrow \\ =2\cdot\#\text{edges}^n}}{\lambda}^{\overset{\substack{\text{random weight from } [2n^2] \\ \text{for Isolation Lemma} \\ \downarrow}}{c(i,j)}} \sum_S x^{f(S)} \\ & & \\ & A_{j,i} = -A_{i,j} & \end{bmatrix}$$

$$\mathrm{pf}(A) \stackrel{\text{def}}{=} \sum \left\{ \operatorname{sgn} \mathcal{M} \cdot \prod_{(i,j) \in \mathcal{M}} A_{i,j} \;\middle|\; \mathcal{M} \text{ perfect matching} \right\}$$

A is a matrix of univariate polynomials of degree $O(k2^k)$ and $O(n^3)$-bit coefficients
$\implies$ pf(A) can be computed in $O^*(2^k)$ time

Output: is coefficient at $x^{k2^k + (2^k - 1)}$ in pf(A) nonzero?