Connectivity Oracles for Predictable Vertex Failures

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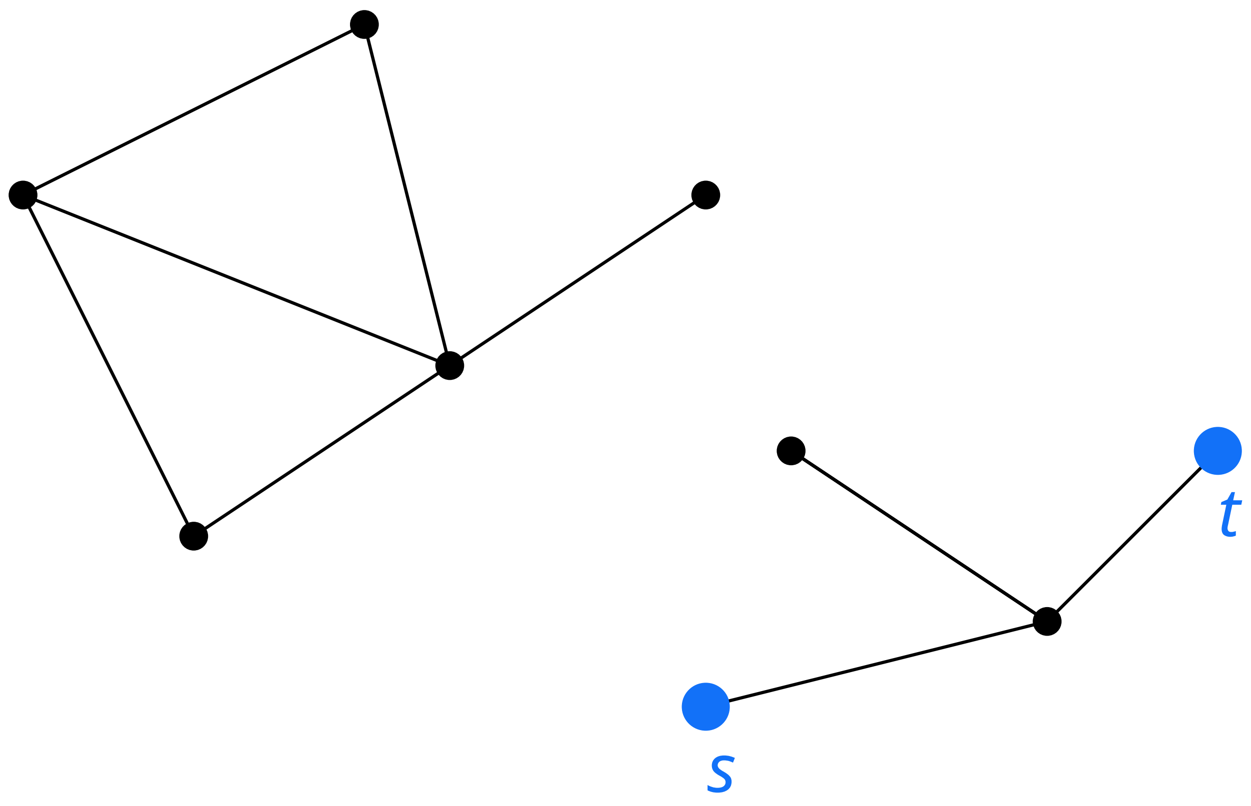
Connectivity oracles

“Are *s* and *t* **connected** in *G*?”



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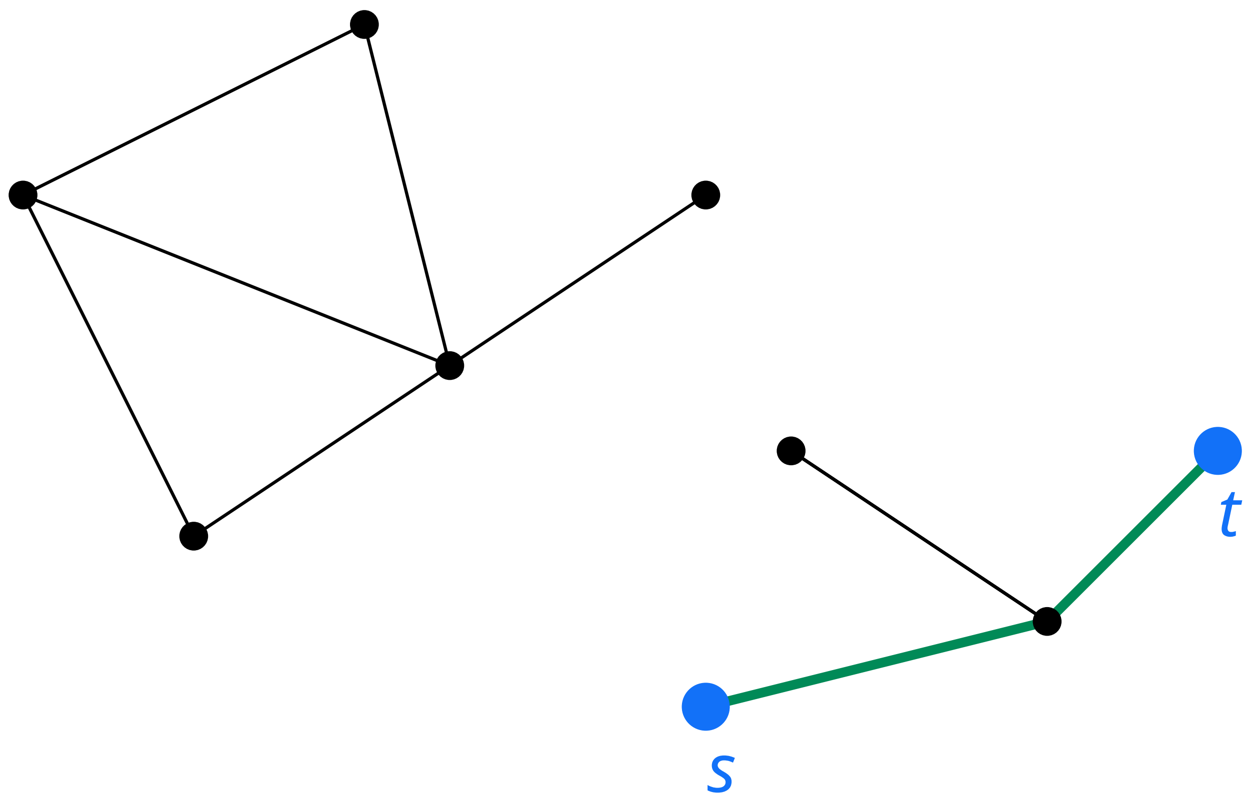

Yes

Connectivity oracles

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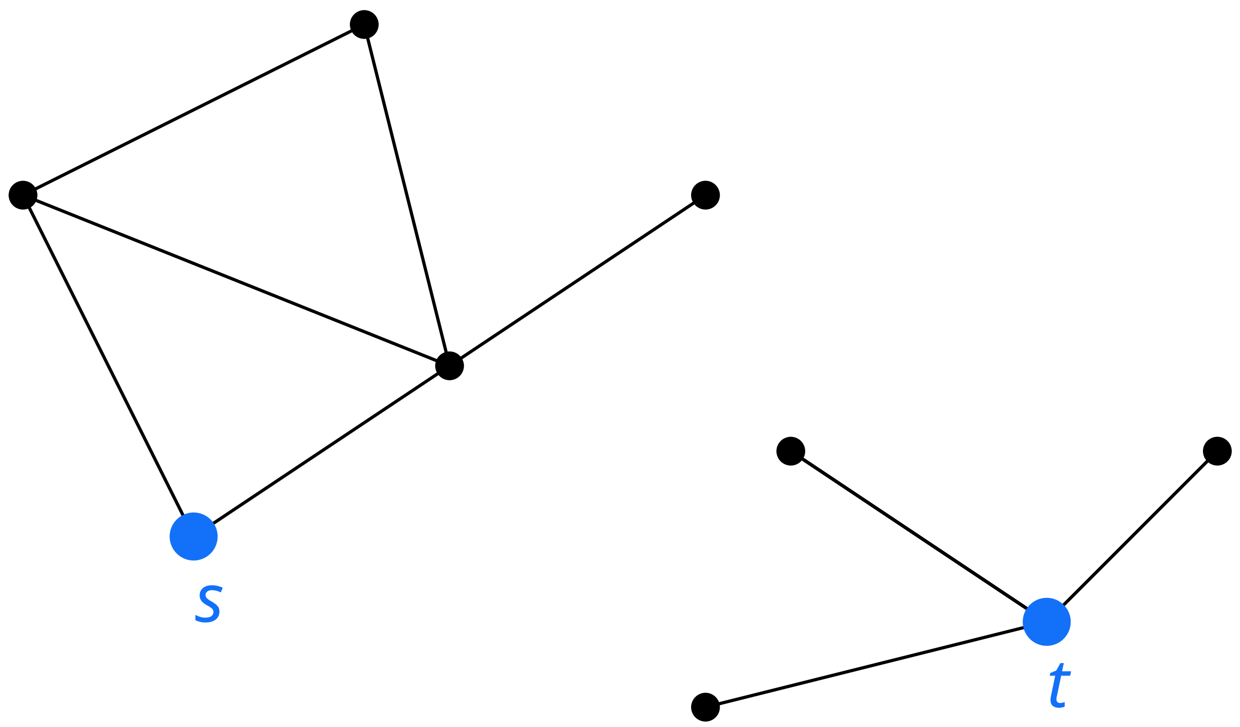
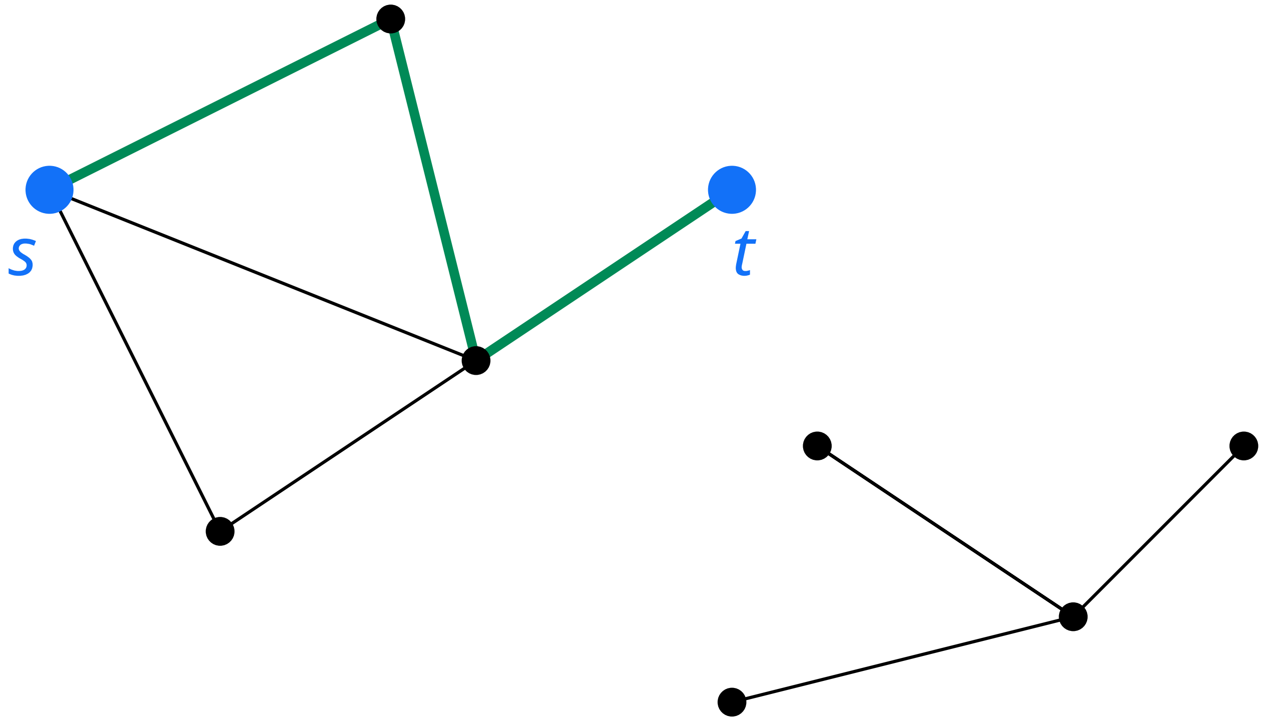


No

Connectivity oracles

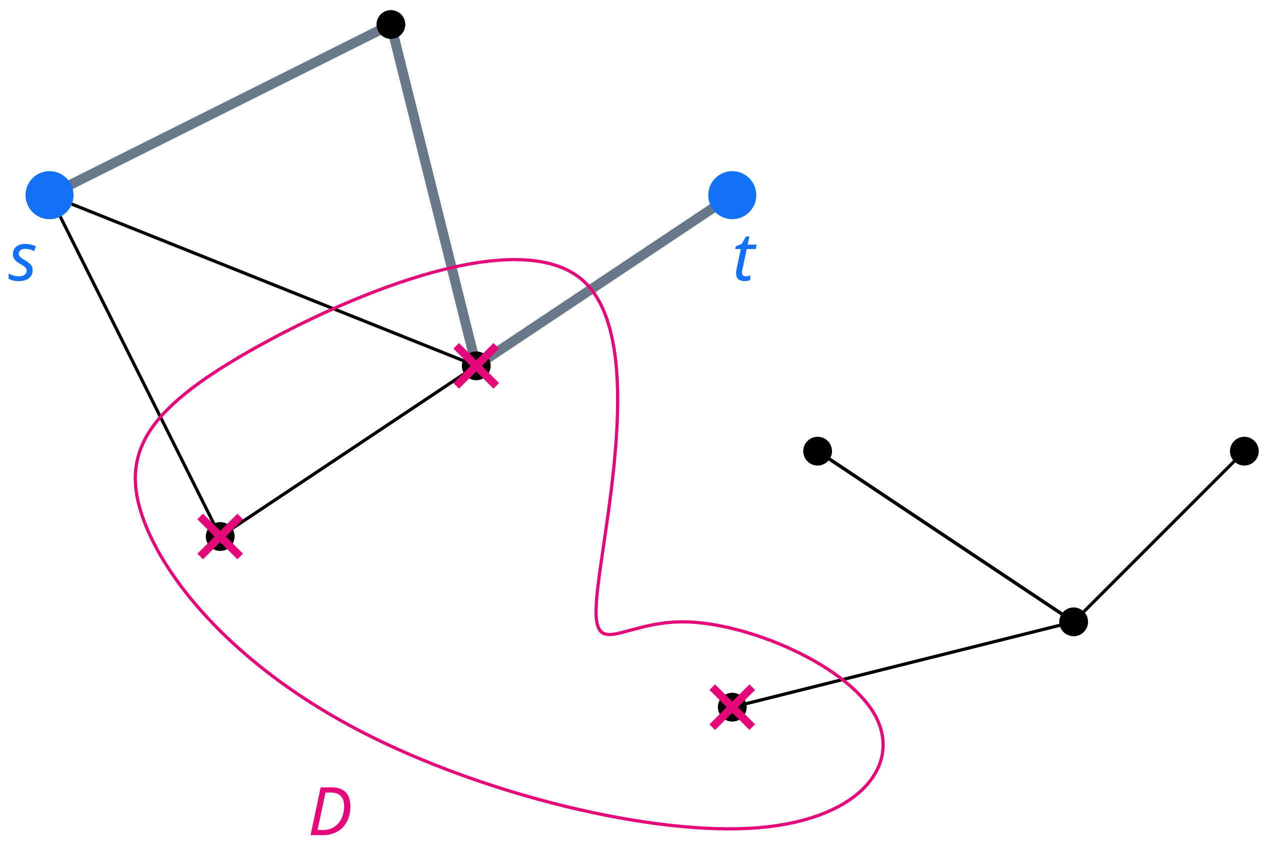
"Are *s* and *t* **connected** in *G*?"



Yes

Connectivity oracles for vertex failures

"Are s and t **connected** in $G \setminus D$?"



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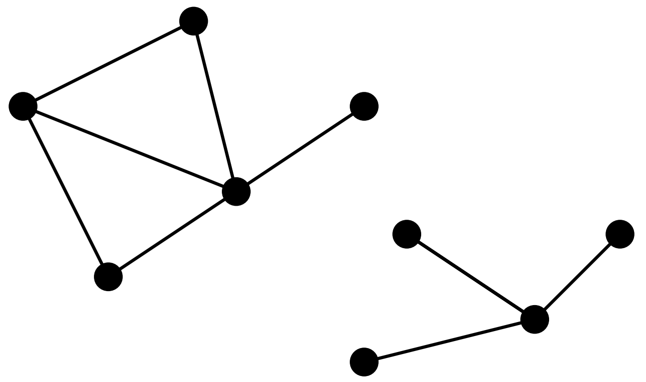
Connectivity oracles for vertex failures

PREPROCESSING



$G = (V, E)$

$|V| = n, |E| = m$

Connectivity oracles for vertex failures

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UPDATE

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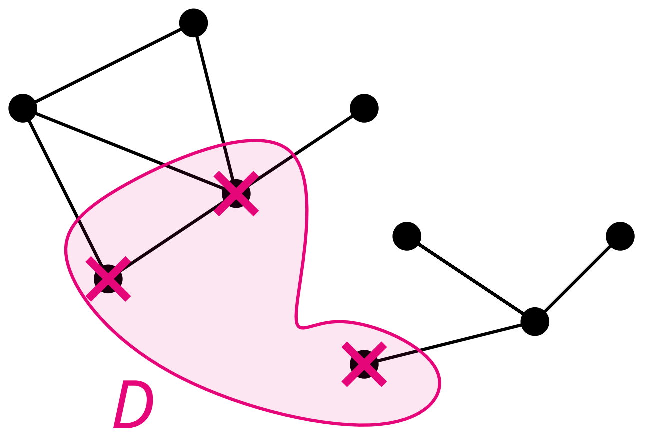
$|D| \leqslant d$



Connectivity oracles for vertex failures

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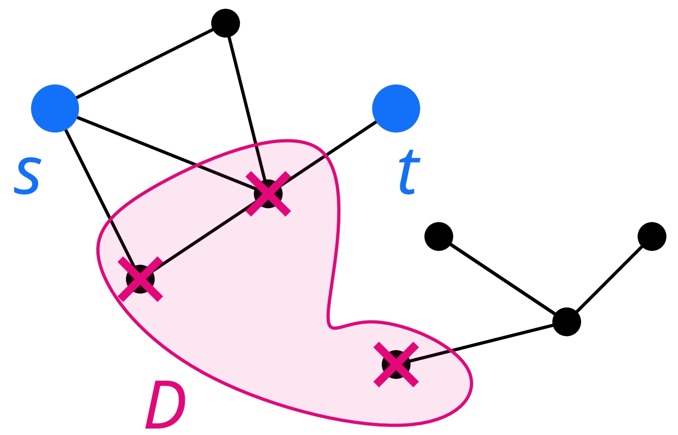

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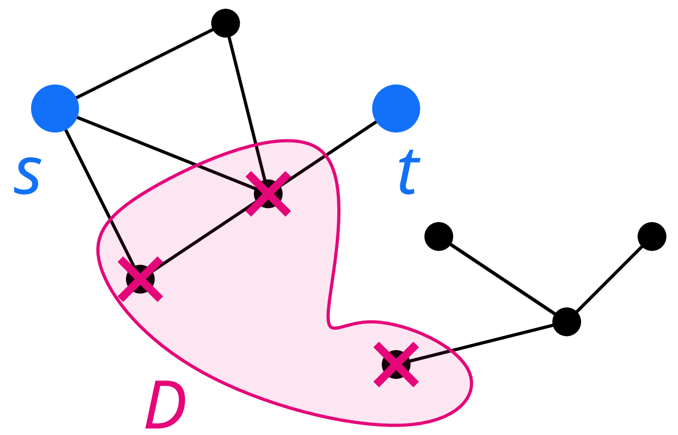
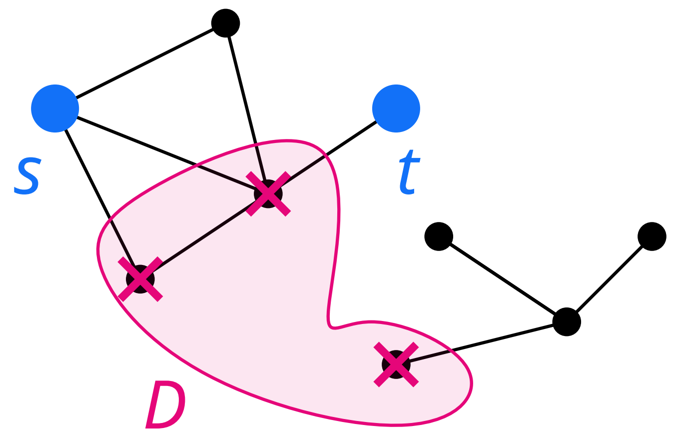


[Duan–Pettie, STOC'10]

Connectivity oracles for vertex failures



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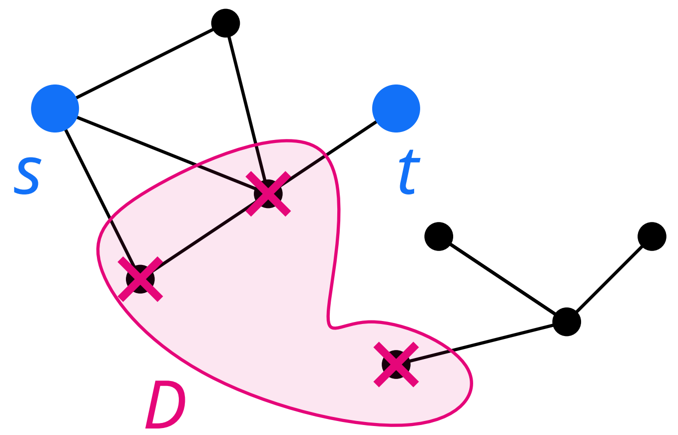
[Duan–Pettie, STOC'10]

$\sum_{u \in D} \deg_u \geqslant \mathrm{poly}(d)$

$\#$ new CCs $\geqslant \mathrm{poly}(d)$

Connectivity oracles for vertex failures



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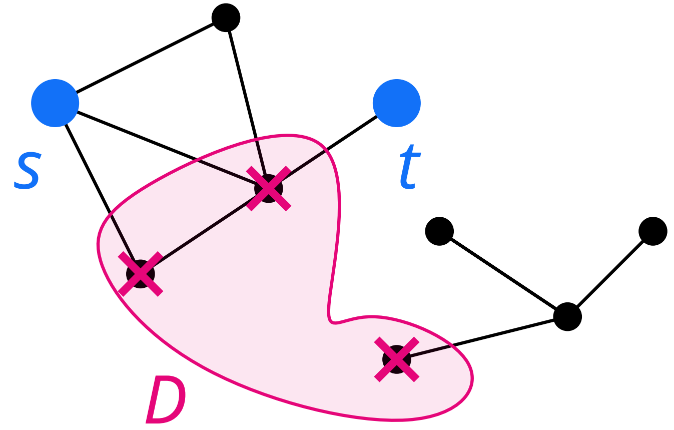
poly(n)	poly(d)	$d^{1-o(1)}$	$\leftarrow$ lower bound	[Henzinger et al., STOC'15]

Connectivity oracles for vertex failures



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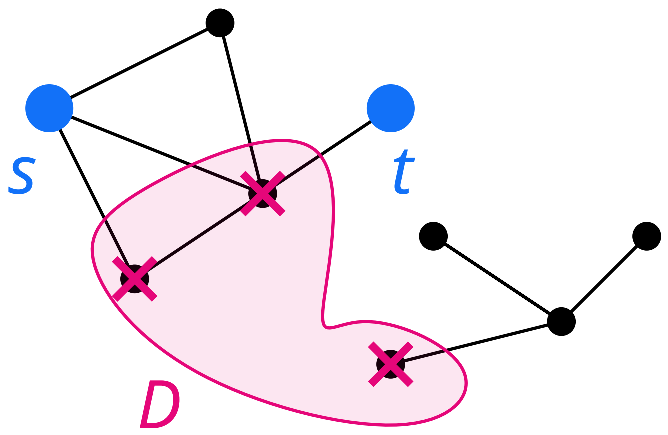

Also tight!

Connectivity oracles for vertex failures

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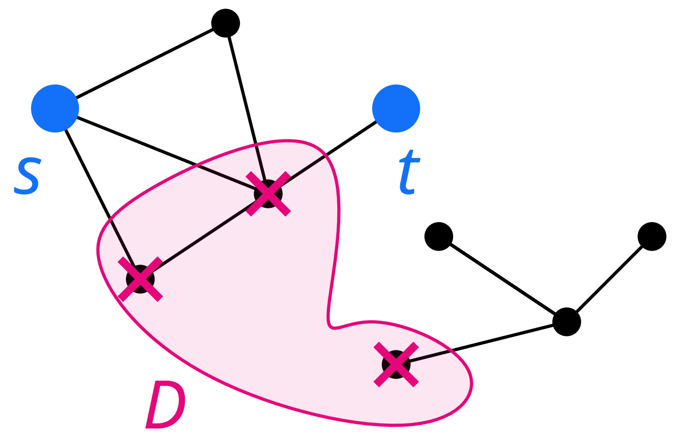

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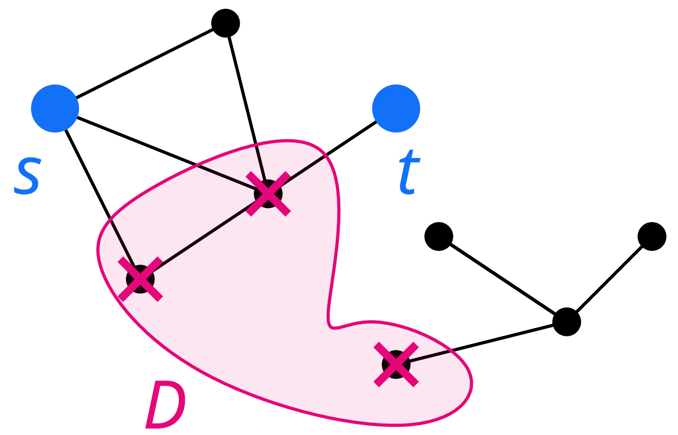

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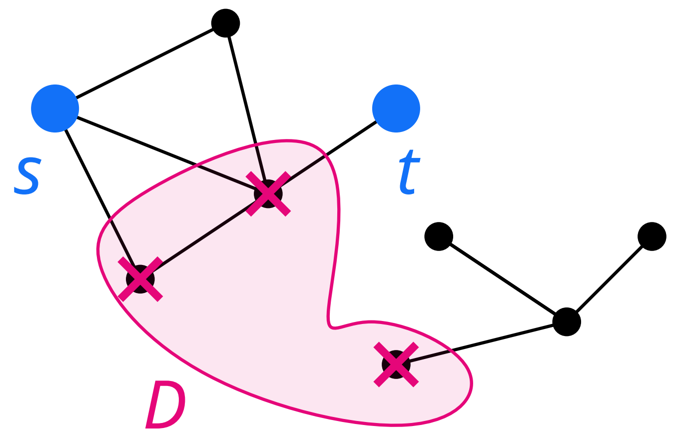


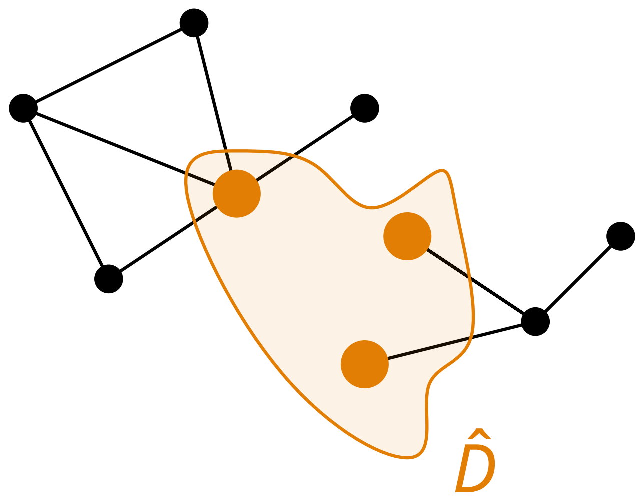
Can we beat the lower bound if we roughly predict which vertices are going to fail?

Connectivity oracles for predictable vertex failures

PREPROCESSING

$G = (V, E)$, $\hat{D} \subseteq V$

$|V| = n$, $|E| = m$



Connectivity oracles for predictable vertex failures

PREPROCESSING

$G = (V, E), \hat{D} \subseteq V$

$|V| = n, |E| = m$

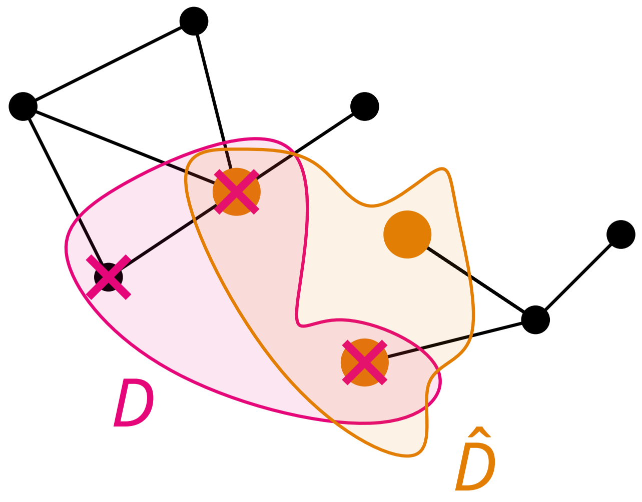
$|\hat{D}| \leqslant d$

UPDATE

$D \subseteq V$

$|D| \leqslant d$

$|\hat{D} \triangle D| \leqslant \eta$ ← prediction error



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QUERY

$s, t \in V \setminus D$

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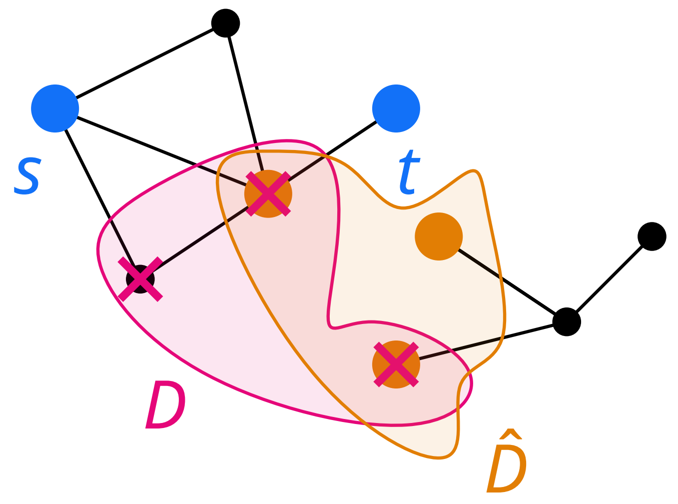

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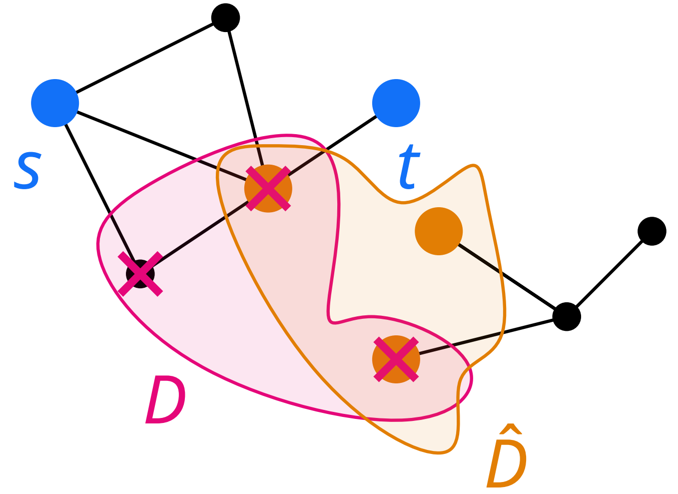
$\widetilde{O}(nm)$	$\widetilde{O}(d^6)$	$O(d)$	[Duan–Pettie, STOC’10]
$\text{poly}(n)$	$\text{poly}(d)$	$d^{1-o(1)}$ ← lower bound	[Henzinger et al., STOC’15]

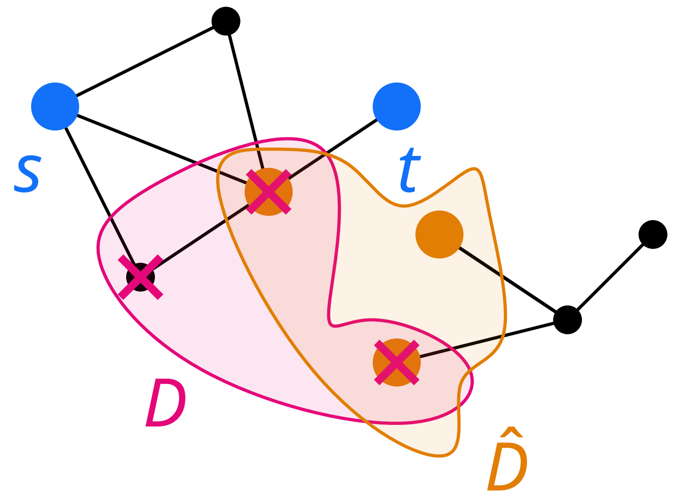
↑ still applies when $\hat{D} = \emptyset, \eta = d$



Goal: query time $\ll d$ (possible only when $\eta \ll d$)

Ideal: query time = $O(\eta)$ (has to be at least $\eta^{1-o(1)}$)

Connectivity oracles for predictable vertex failures



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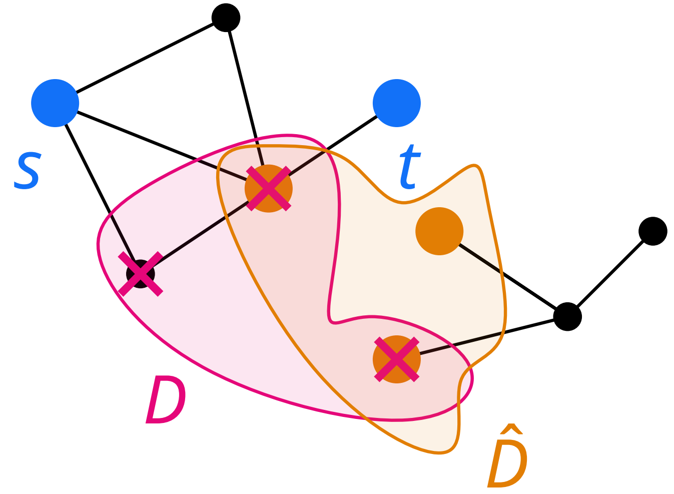
Connectivity oracles for **predictable** vertex failures



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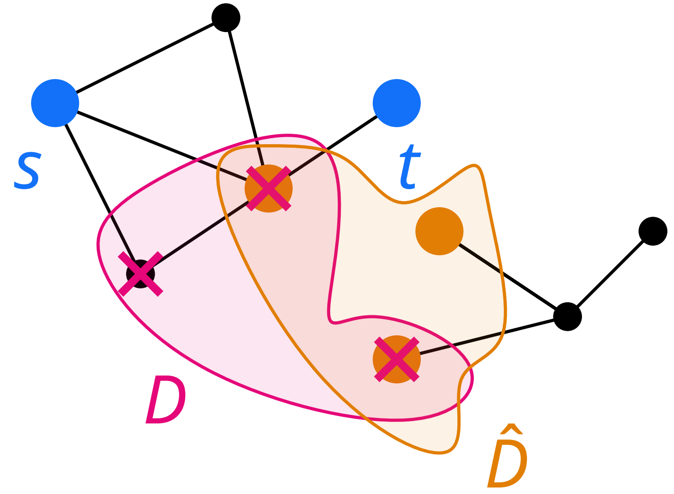
$\widetilde{O}(d^3 m)$	$\widetilde{O}(\eta^4)$	$O(\eta^2)$	[Henzinger–Neumann, ESA'16]

- $\hat{D}$ = initially inactive vertices, $\hat{D} \setminus D$ = reactivations, $D \setminus \hat{D}$ = deactivations

Connectivity oracles for predictable vertex failures



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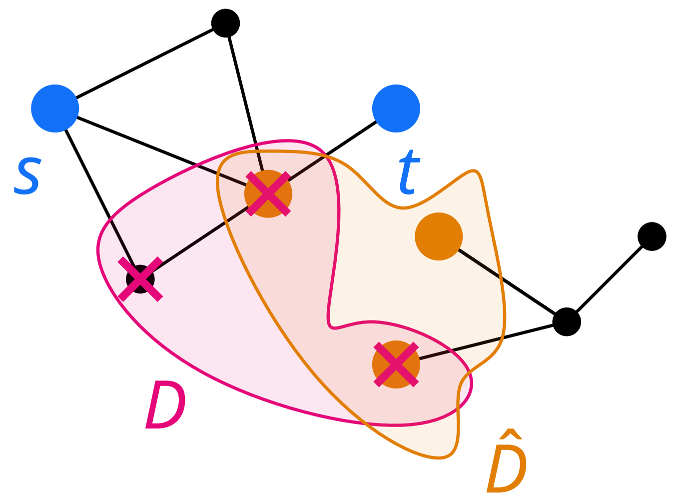
$\widetilde{O}(n^\omega)$	$\widetilde{O}(\eta^\omega)$	$O(\eta^2)$	[van den Brand–Saranurak, FOCS'19]

- directed graphs, edge deletions/insertions can simulate node de-/re-activation

Connectivity oracles for predictable vertex failures



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Technical sneak peek

Recall the last two slides. . .

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$\widetilde{O}(dm)$	$\widetilde{O}(d^2)$	$O(d)$	(deterministic)	[Long–Saranurak, FOCS'22]
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…and connect the dots!

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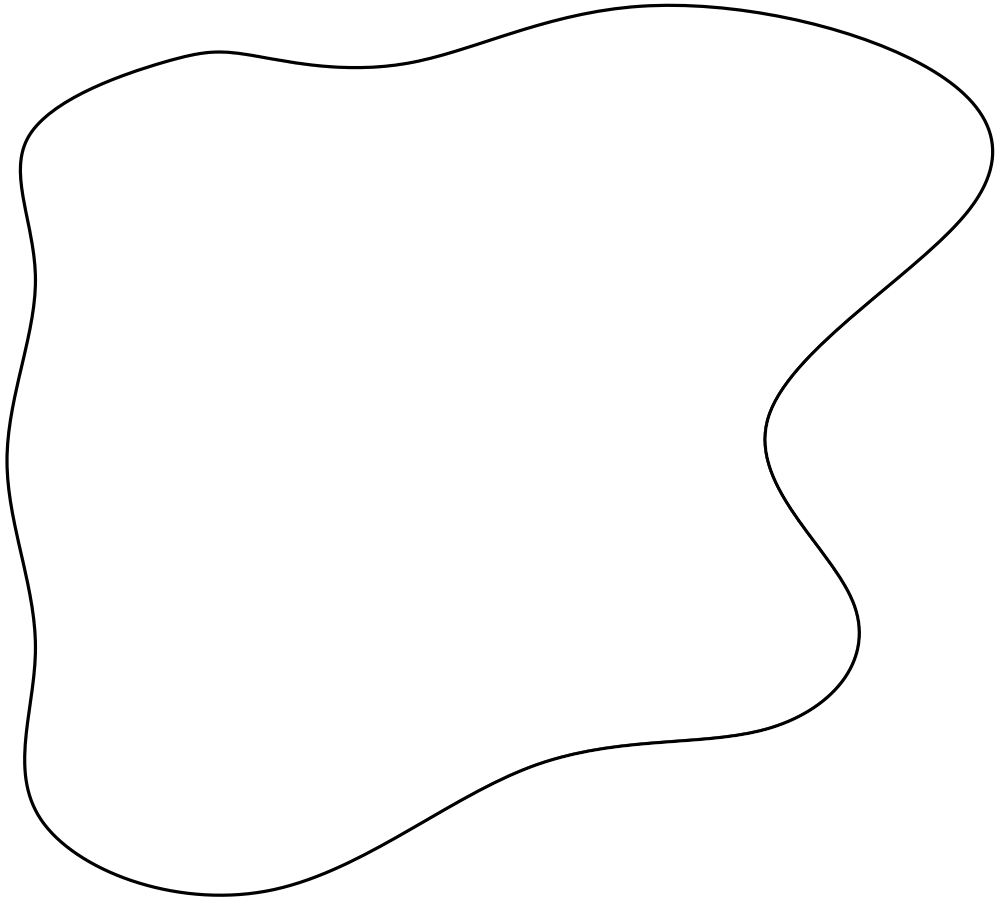
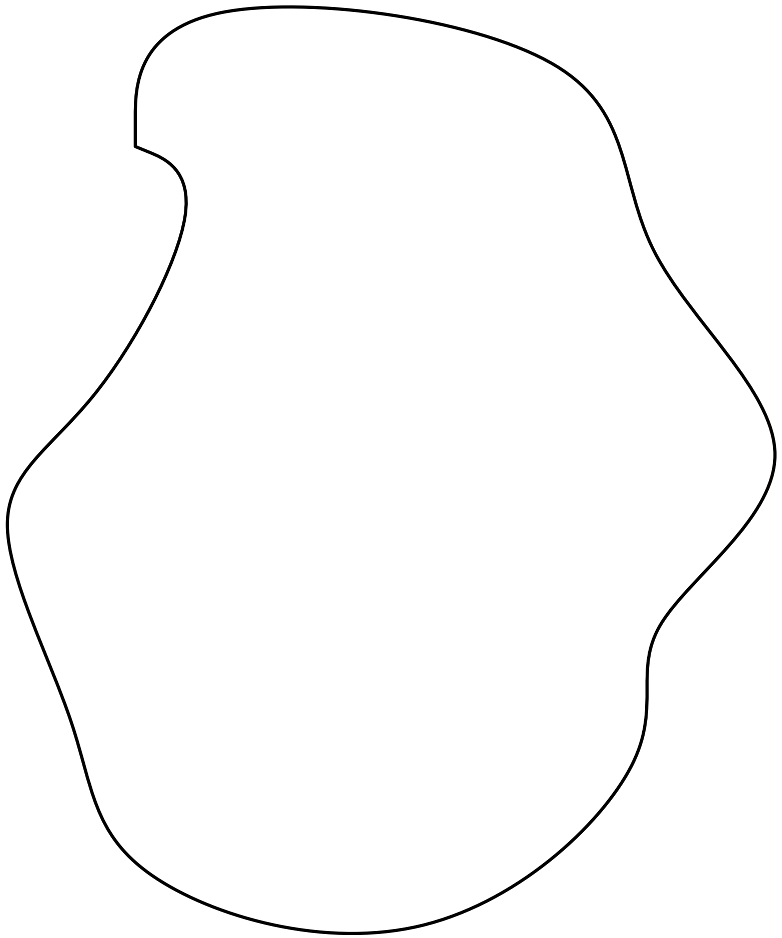
Technical sneak peek, more serious

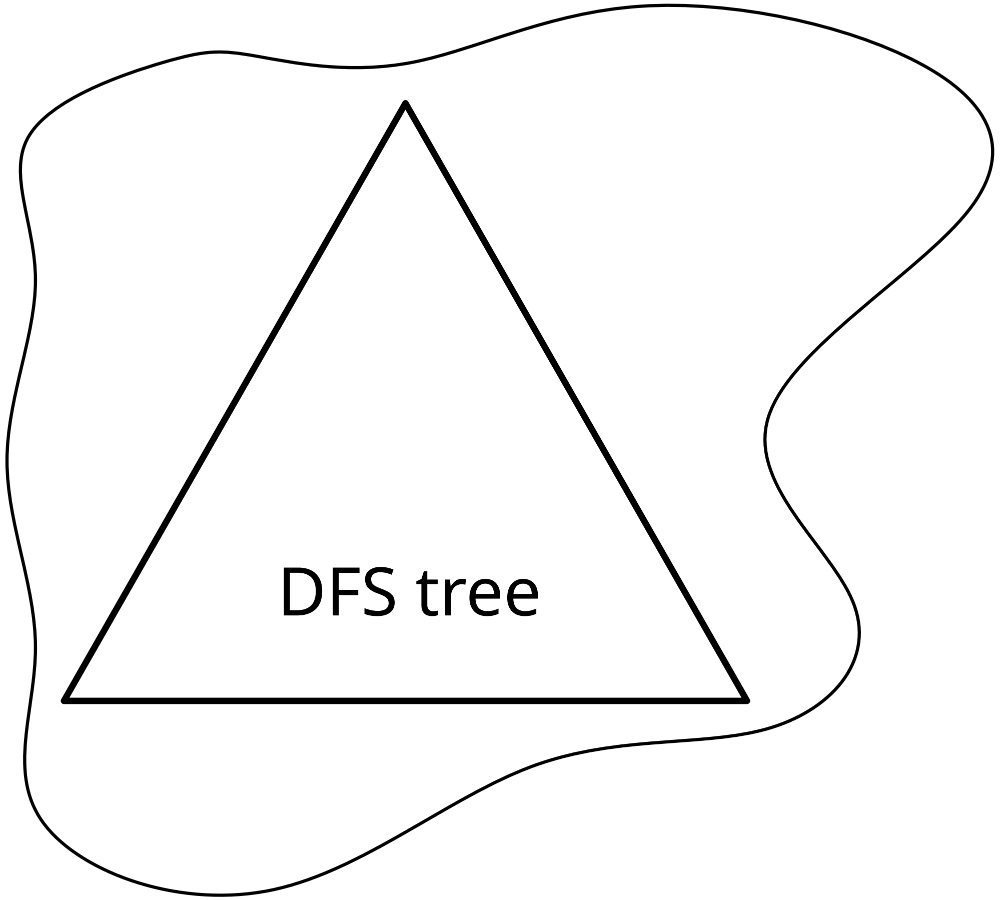


$G \setminus \hat{D}$



$\hat{D}$

Technical sneak peek, more serious



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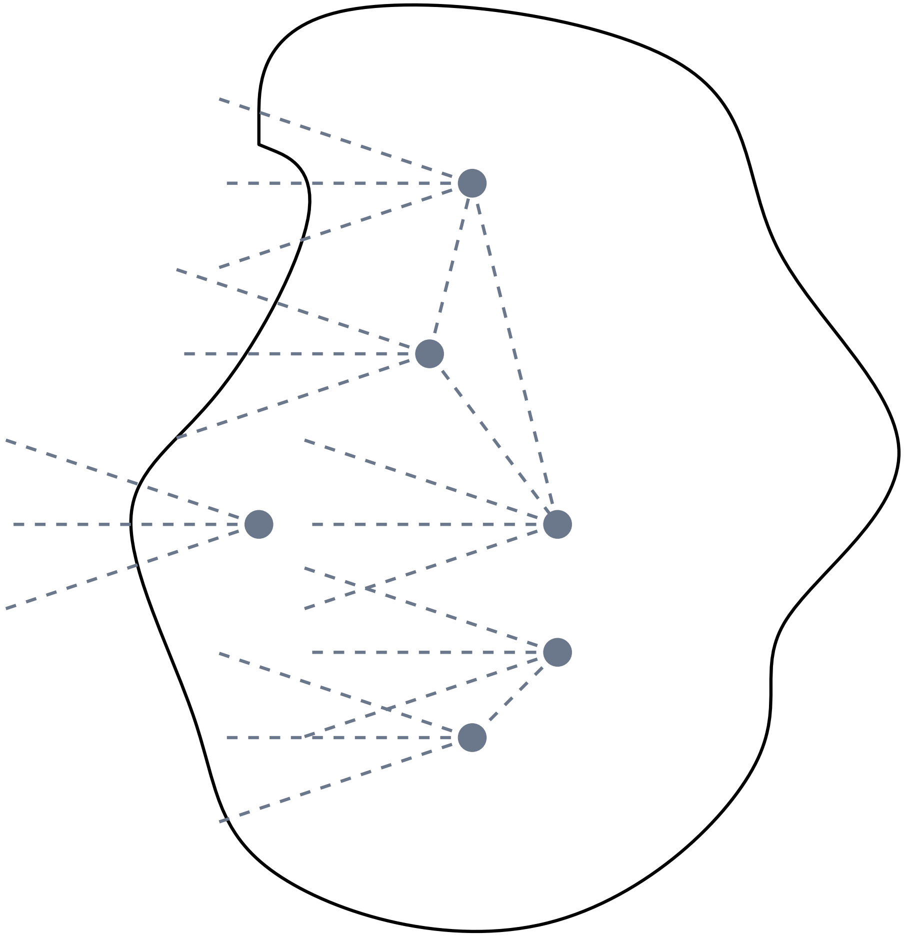

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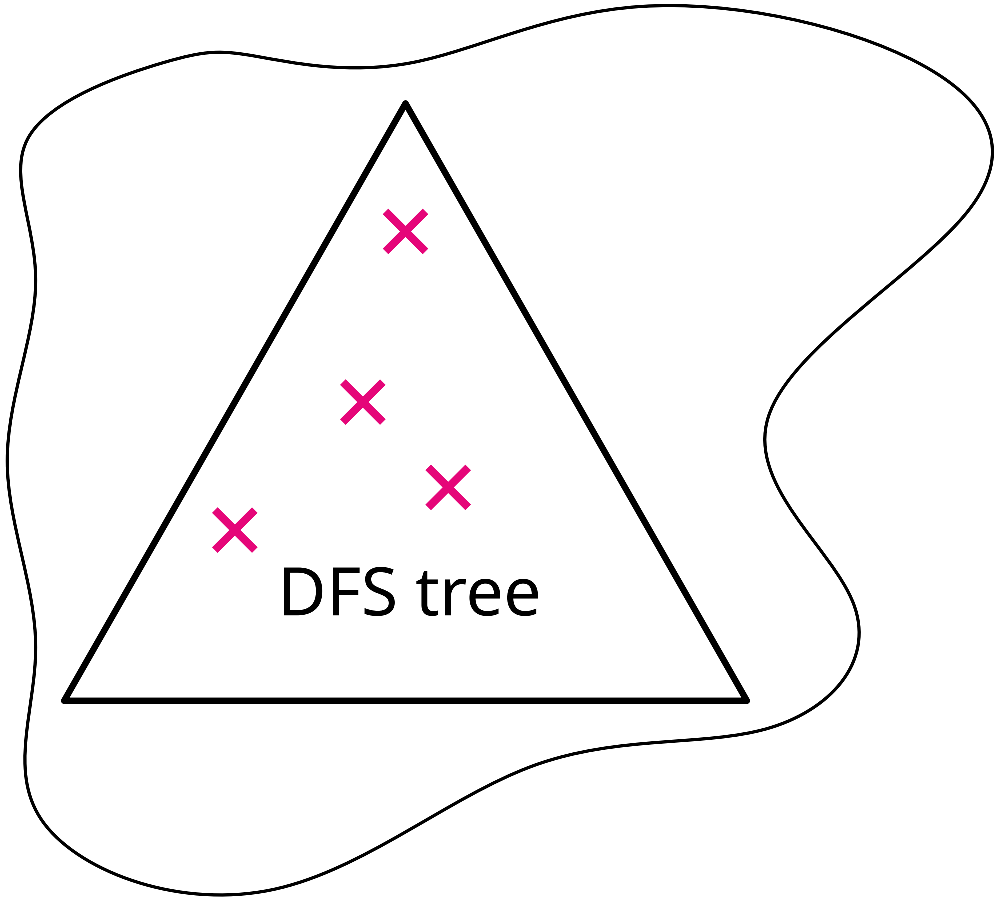

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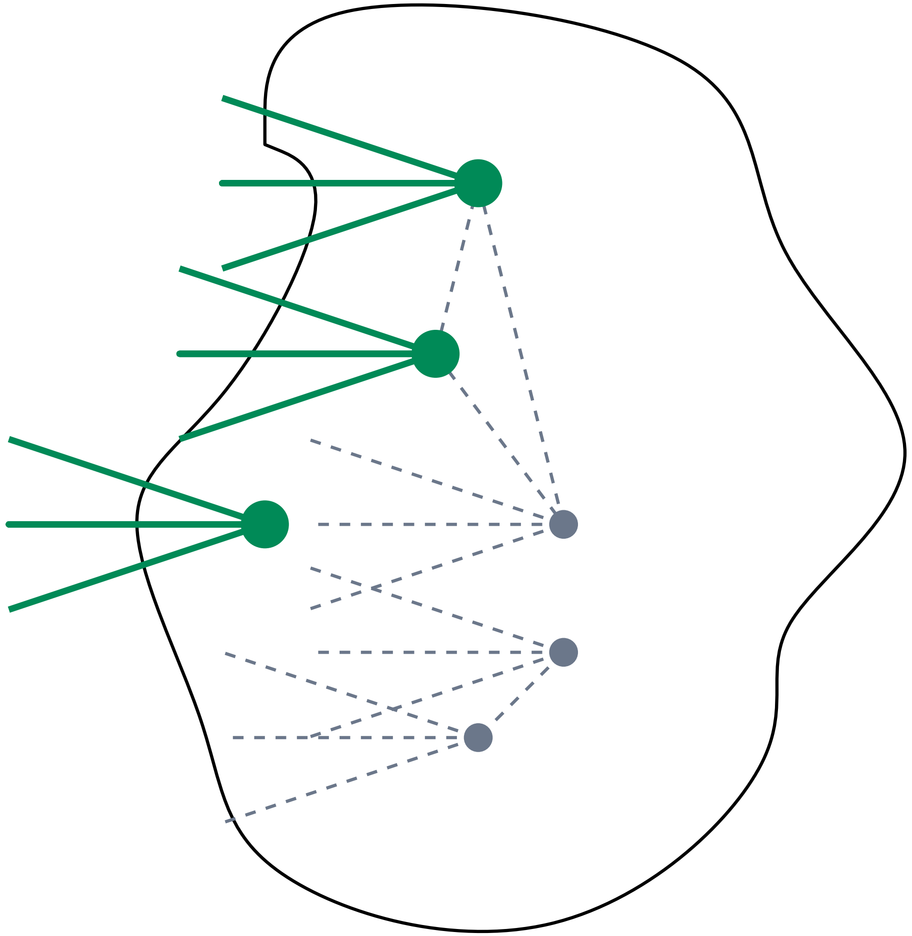

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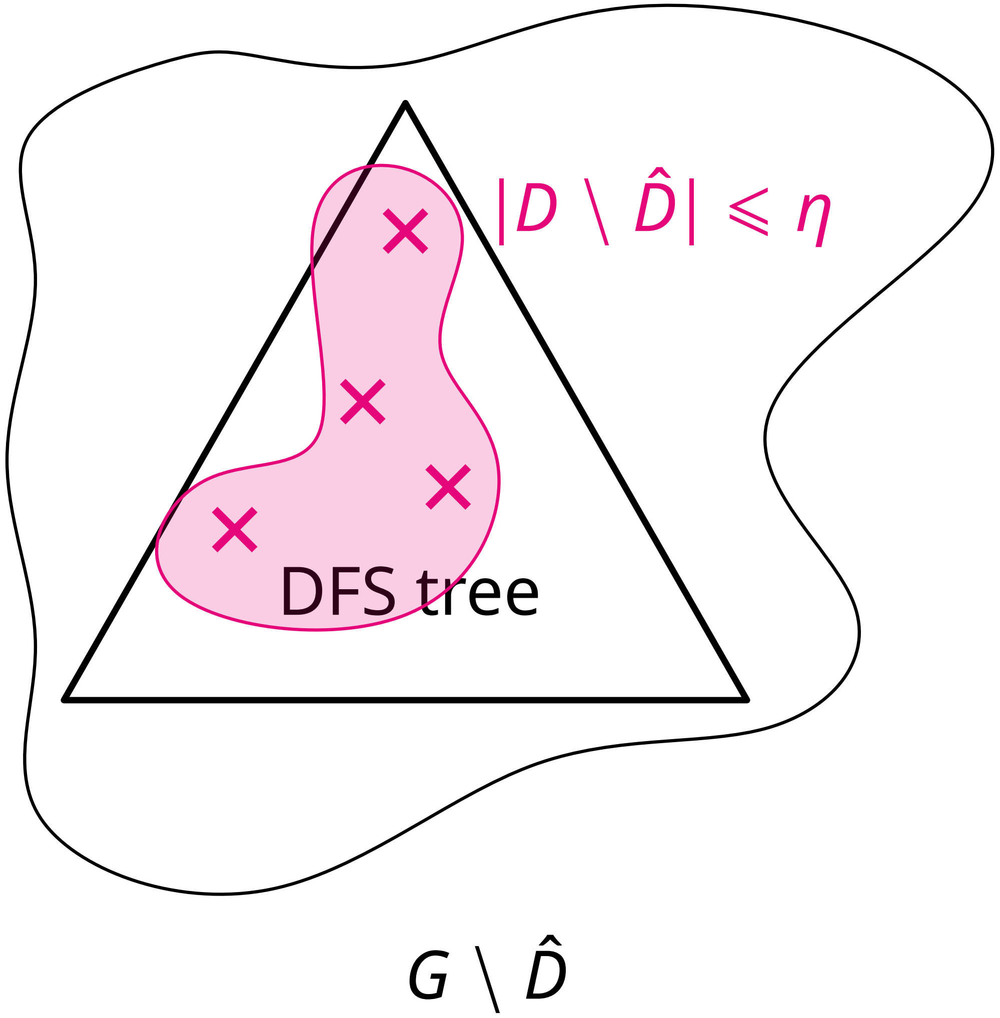
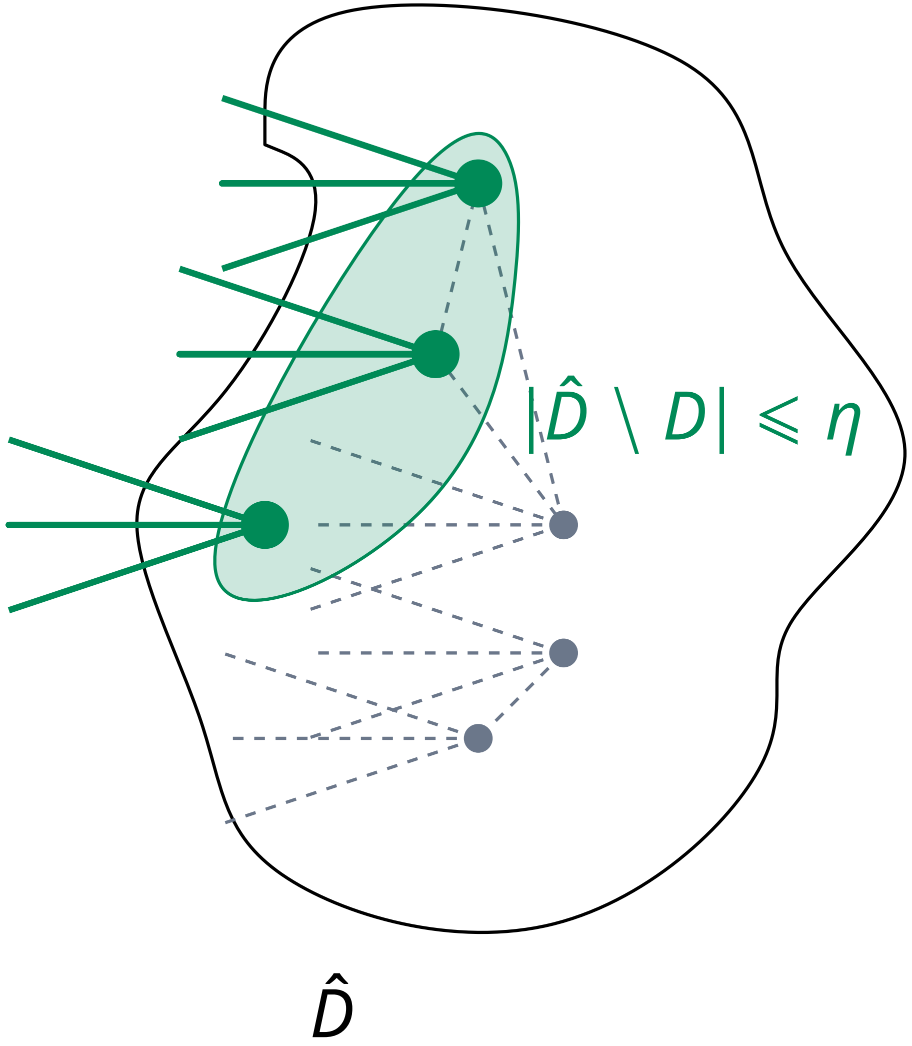
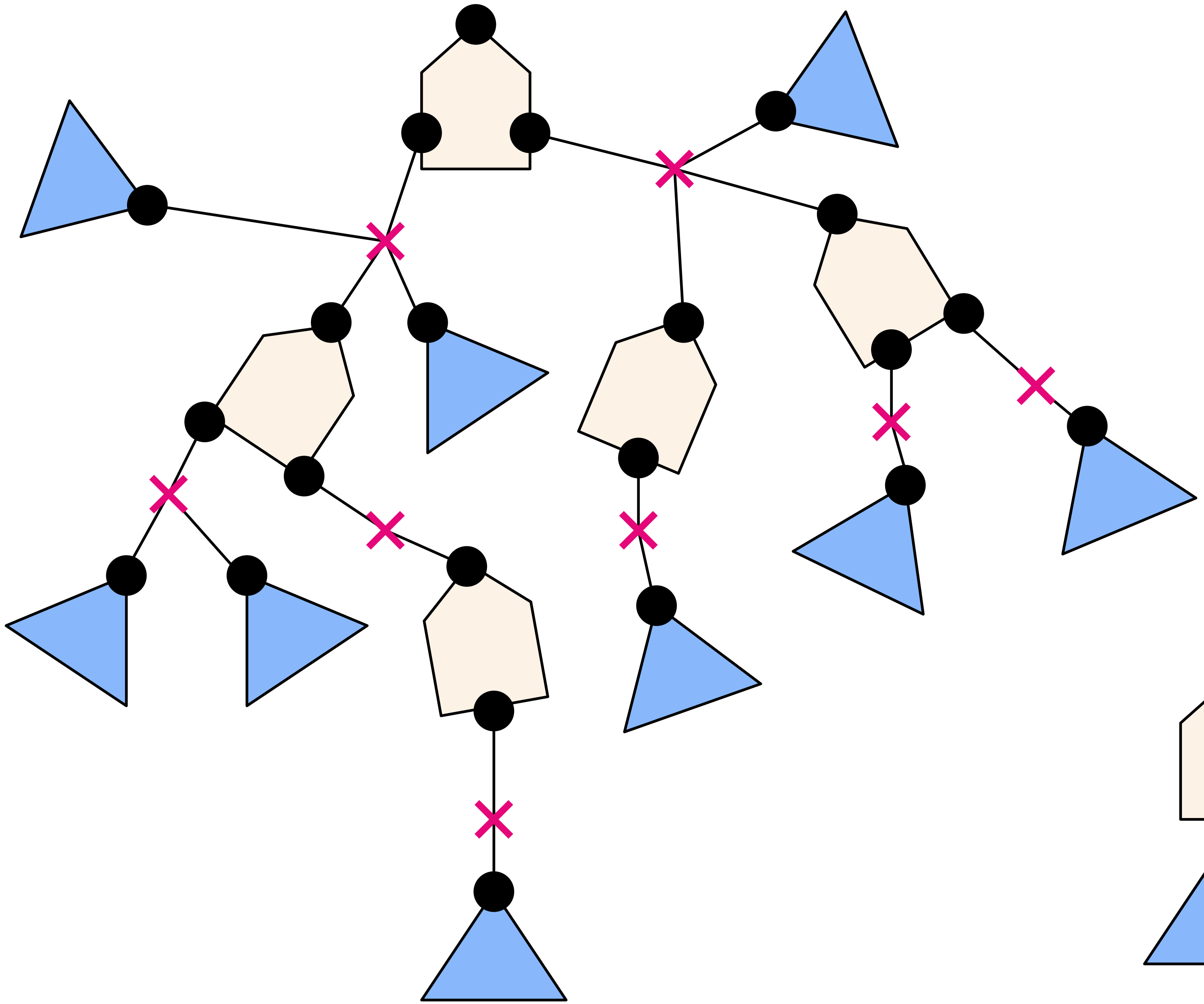
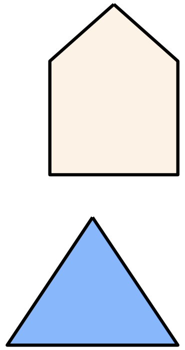




Technical sneak peek, DFS tree decomposition





“internal components”, # $\leqslant \eta$

“hanging subtrees”, plentiful

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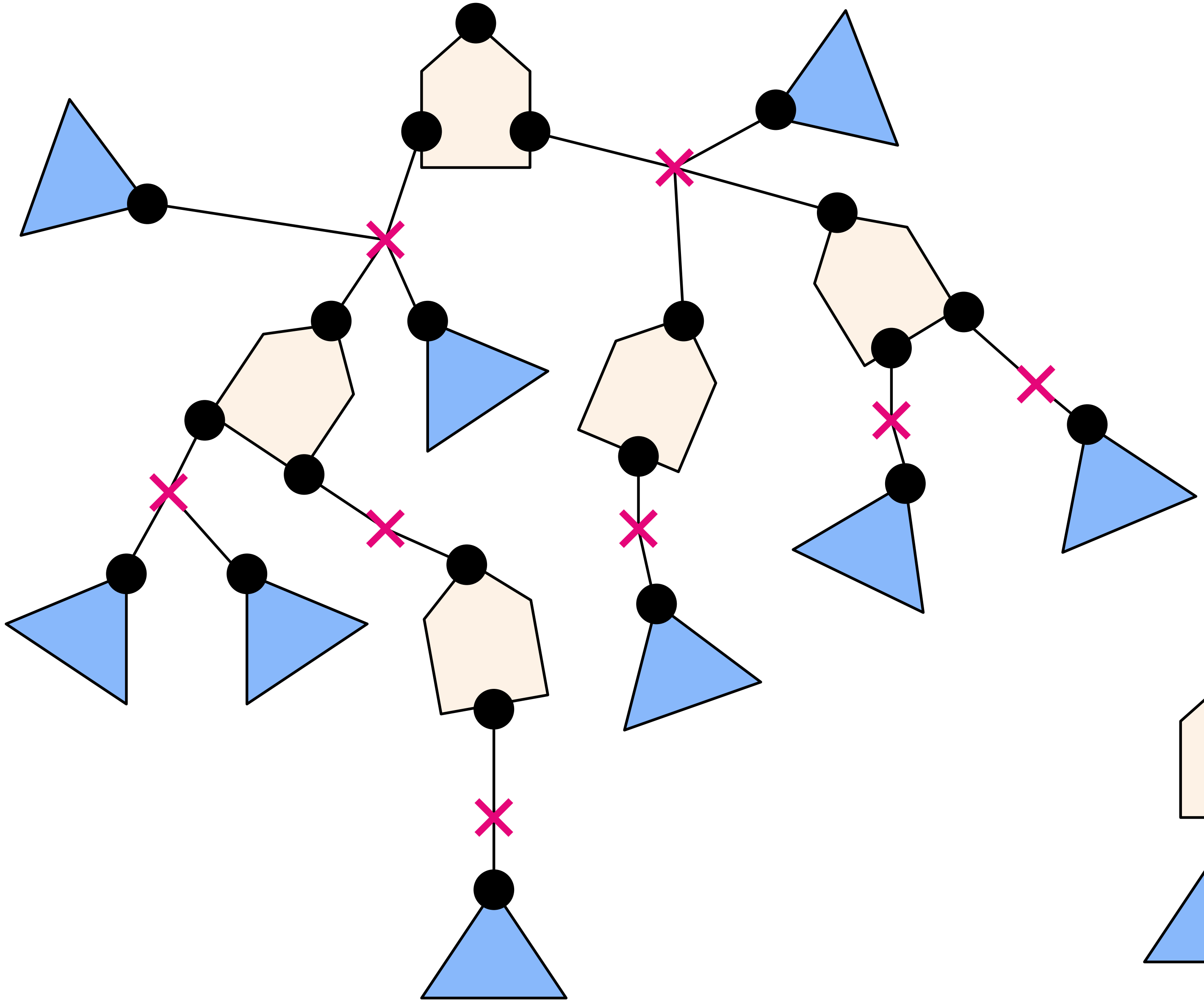

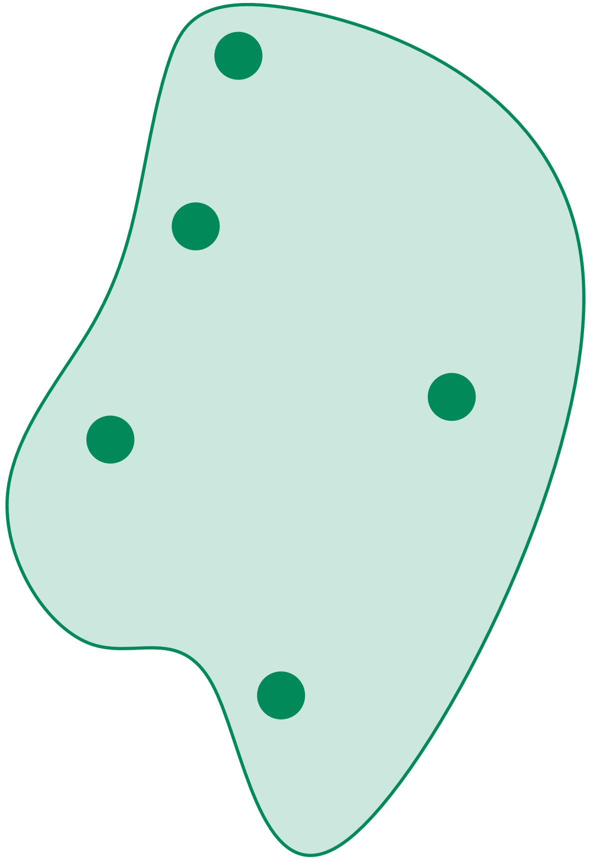

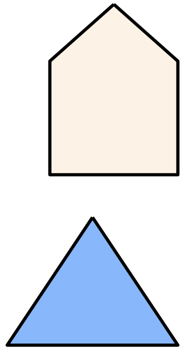

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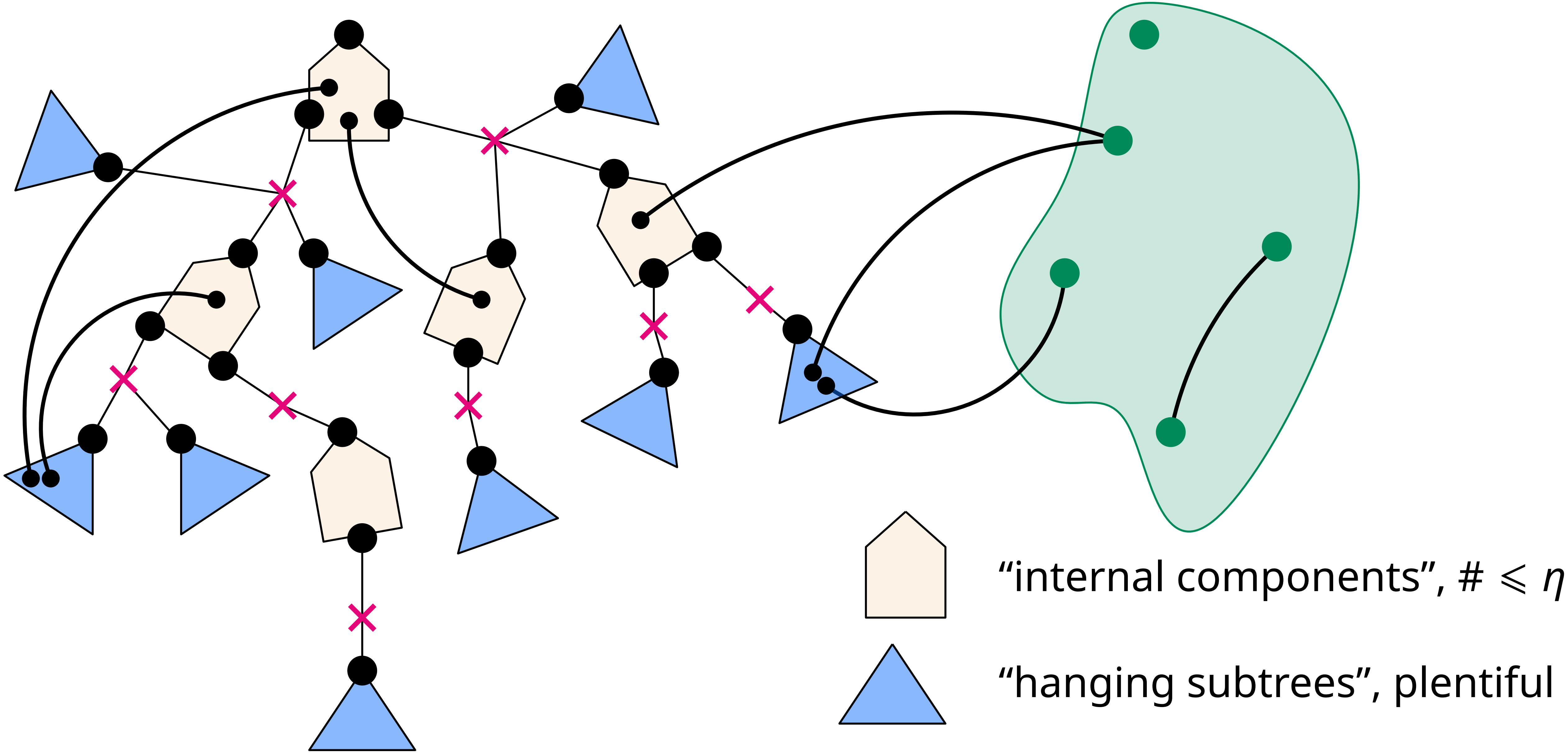

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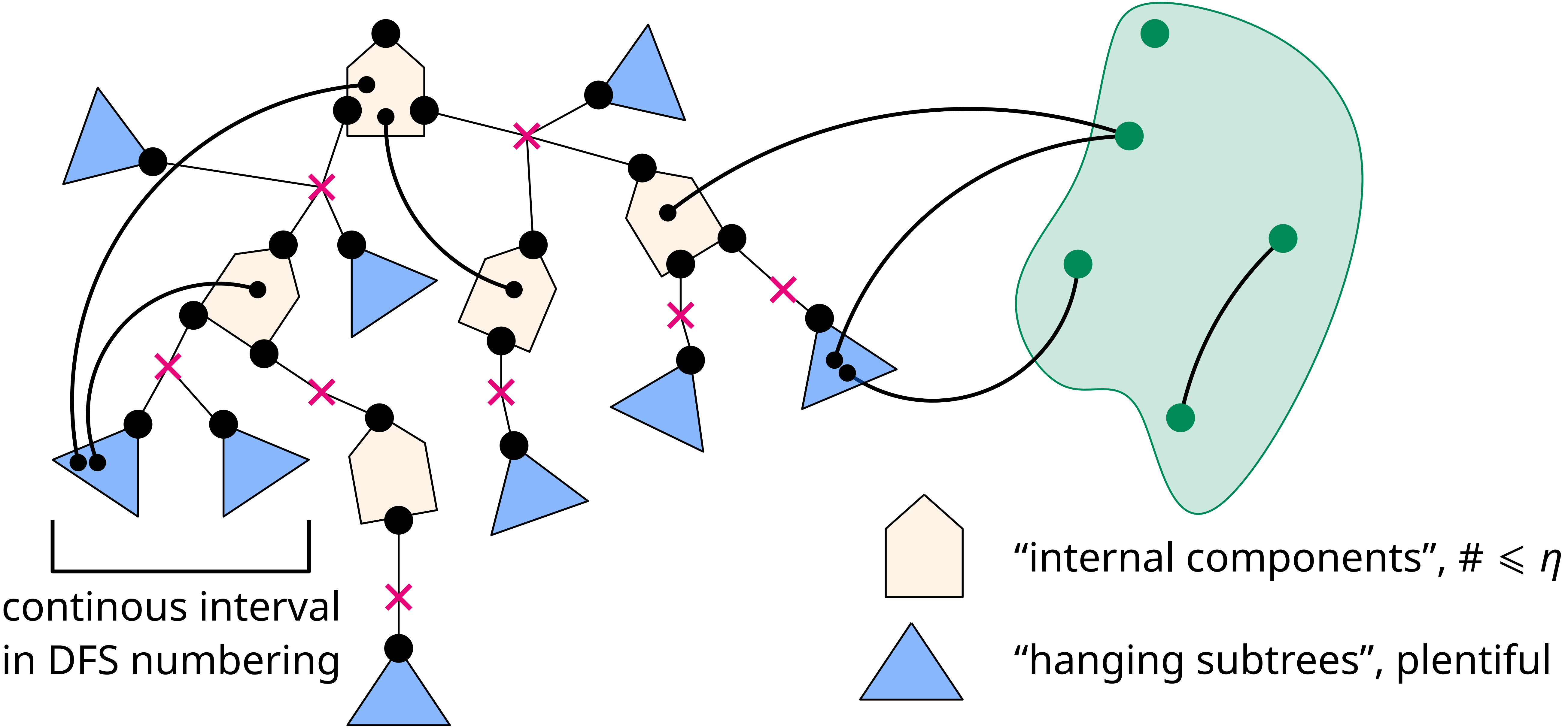
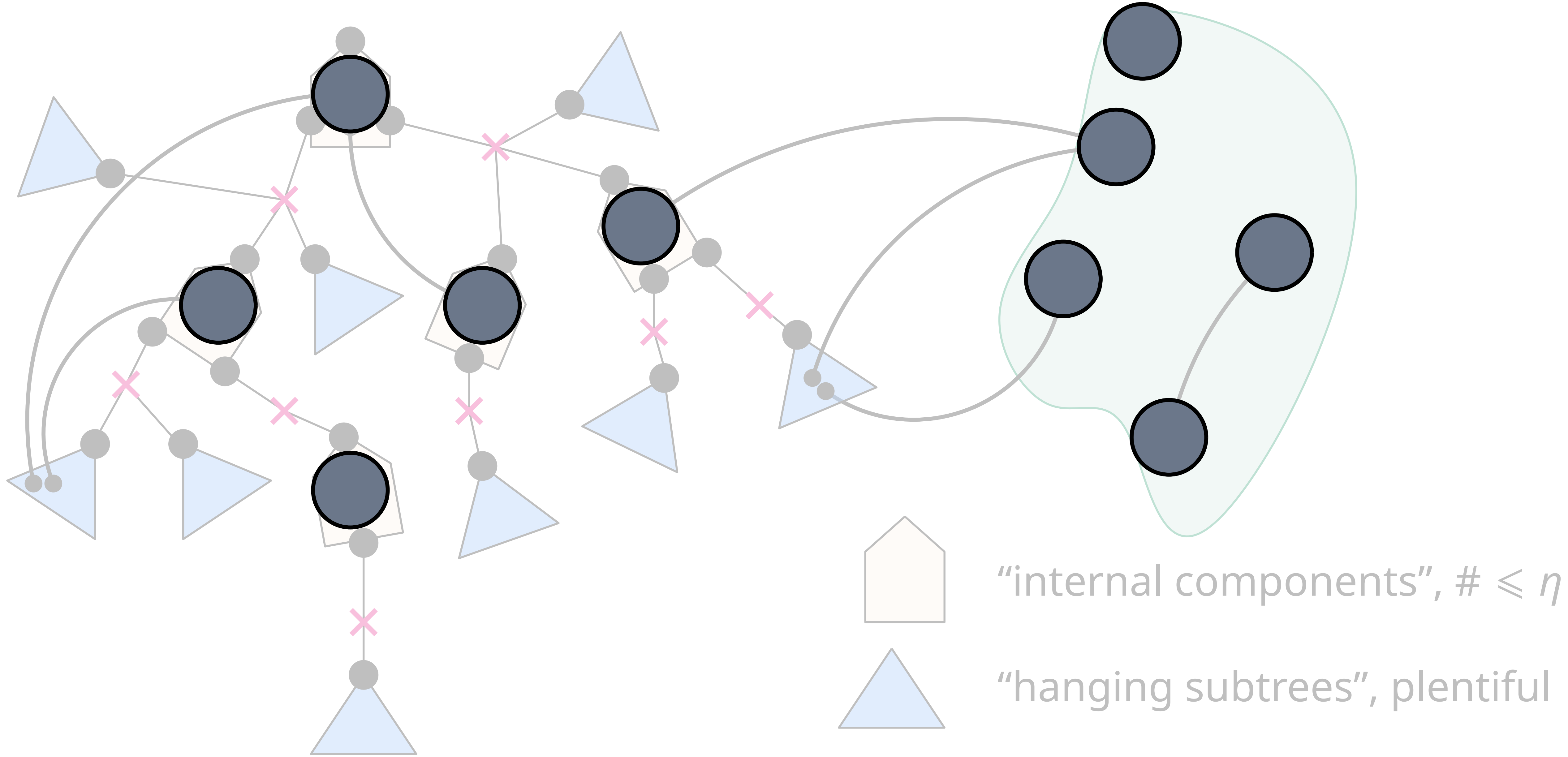
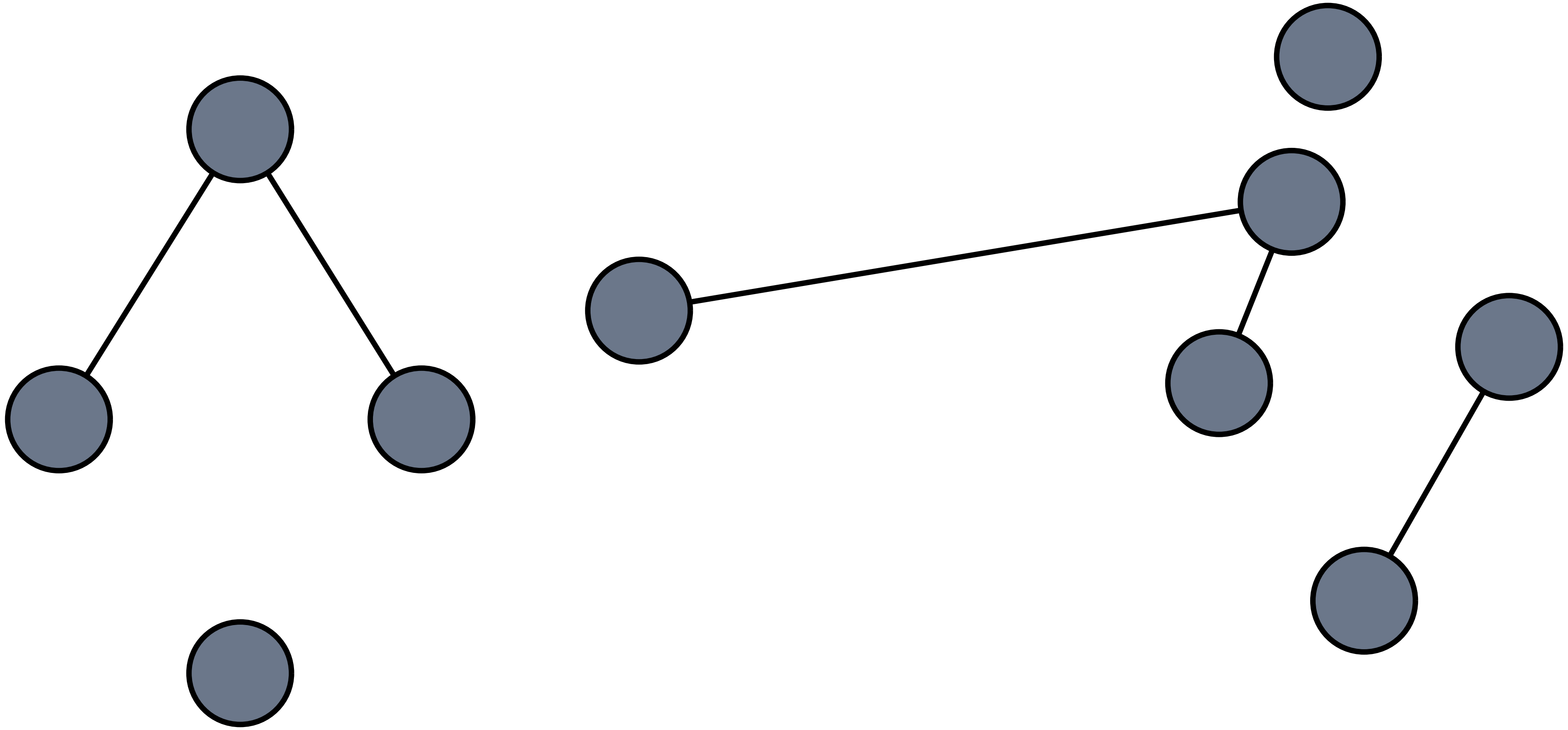


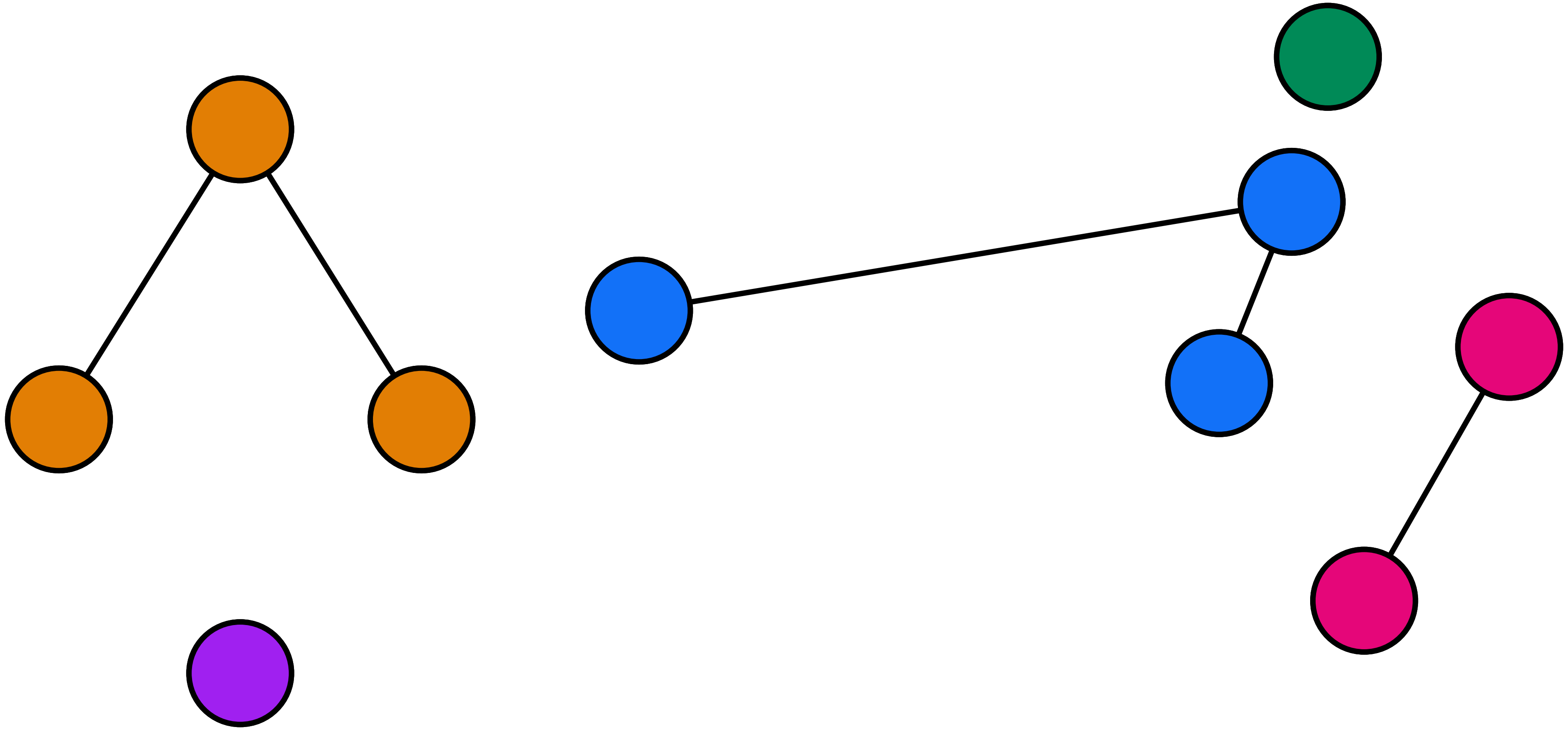
Technical sneak peek, DFS tree decomposition



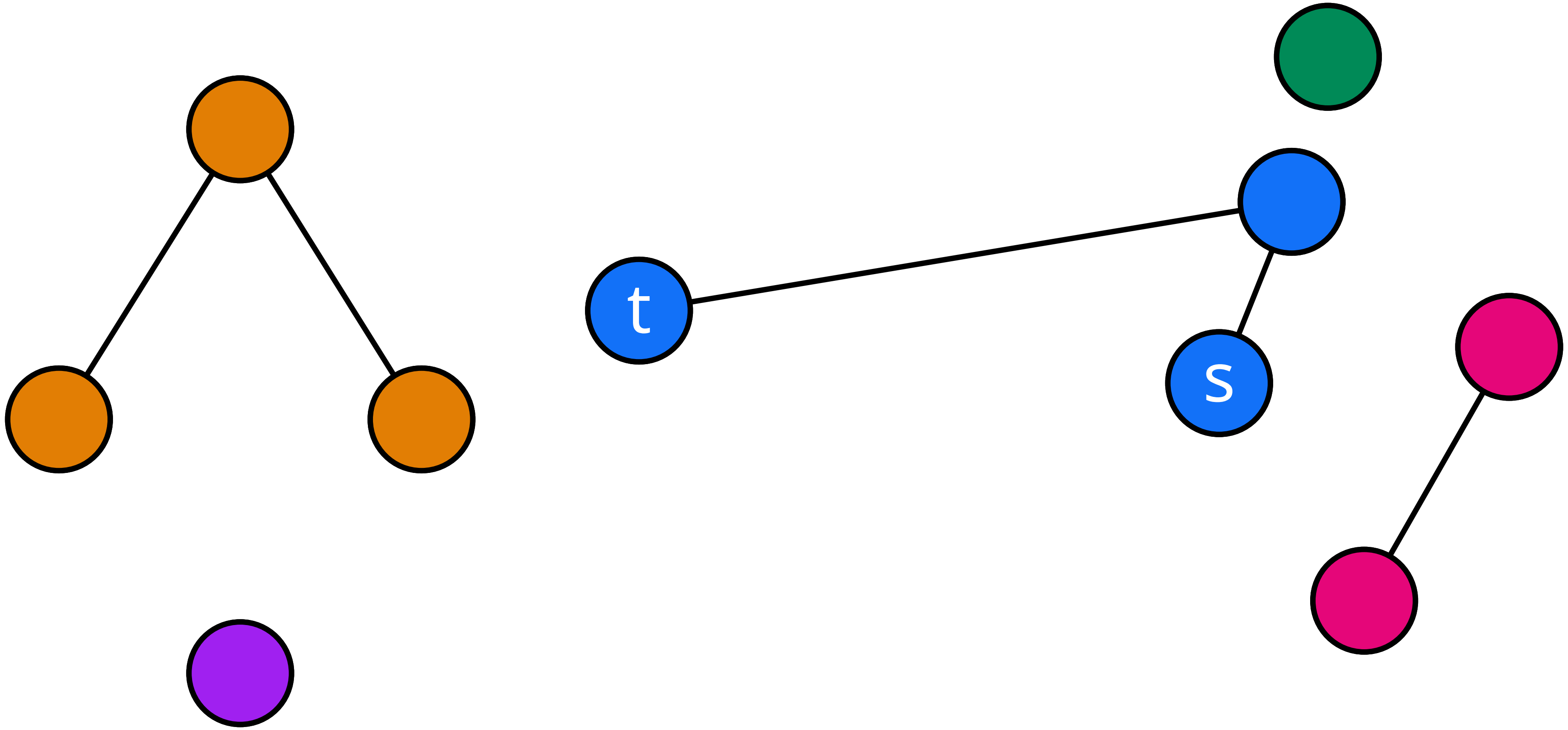
Technical sneak peek, the auxiliary graph



Technical sneak peek, the auxiliary graph



Technical sneak peek, the auxiliary graph



THANK YOU!