Bellman–Ford is optimal

for shortest hop-bounded paths

Tomasz Kociumaka **Adam Polak**

MAX PLANCK INSTITUTE FOR INFORMATICS

Bocconi

Bellman–Ford algorithm

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$d[source] \leftarrow 0;$

for i **from** 1 **to** $n - 1$ **do**

  foreach edge $(u, v) \in E$ **do**

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even in **undirected graphs** with **nonnegative edge weights**.

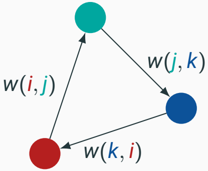
The negative triangle problem

Given (complete) graph on N nodes

with **edge weights w** $: [N] \times [N] \to \mathbb{Z}$

find $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \in [N]$

such that $\boldsymbol{w(i, j) + w(j, k) + w(k, i) < 0}$



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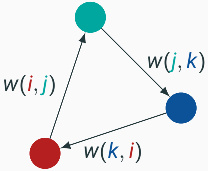
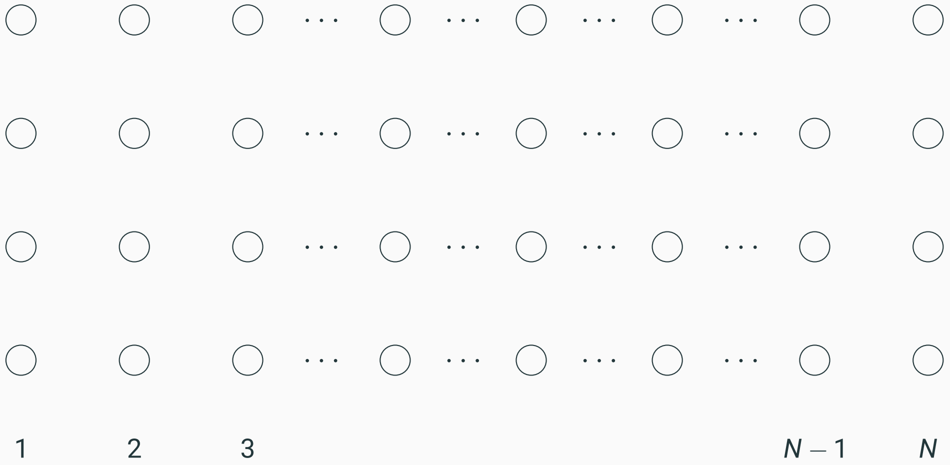


No $O(N^{3-\varepsilon})$ time algorithm unless the APSP hypothesis fails.

[Vassilevska Williams, Williams '10]

Reduction: negative triangle $\longrightarrow$ shortest hop-bounded path



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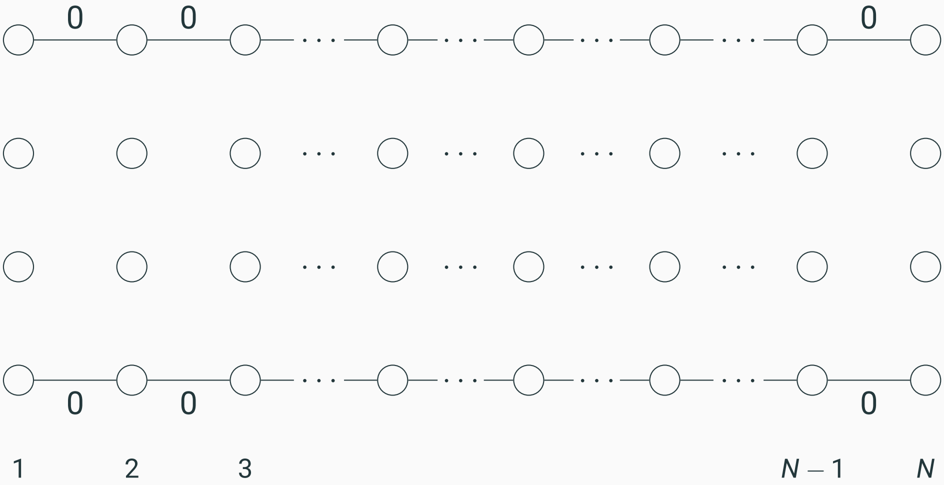

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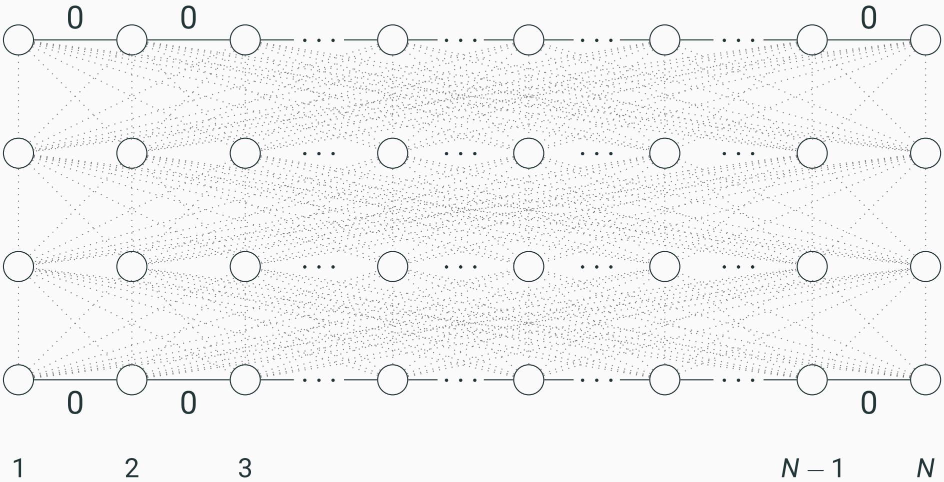

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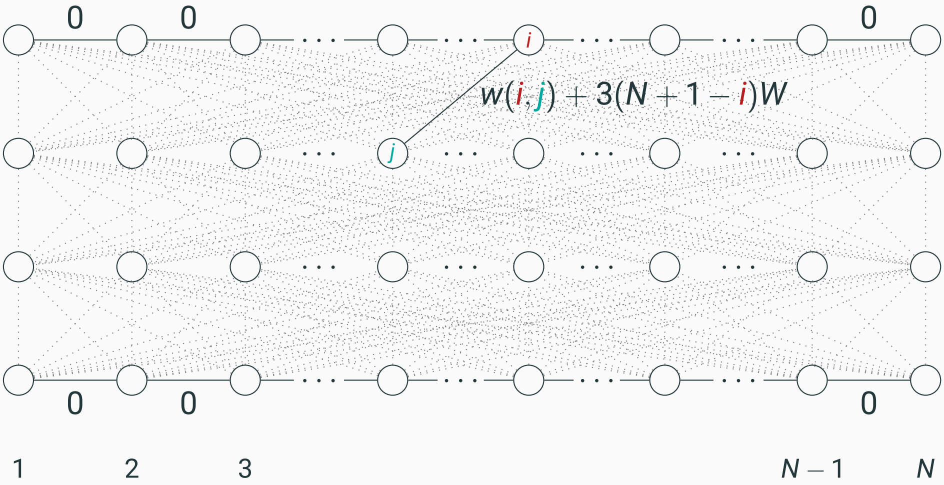

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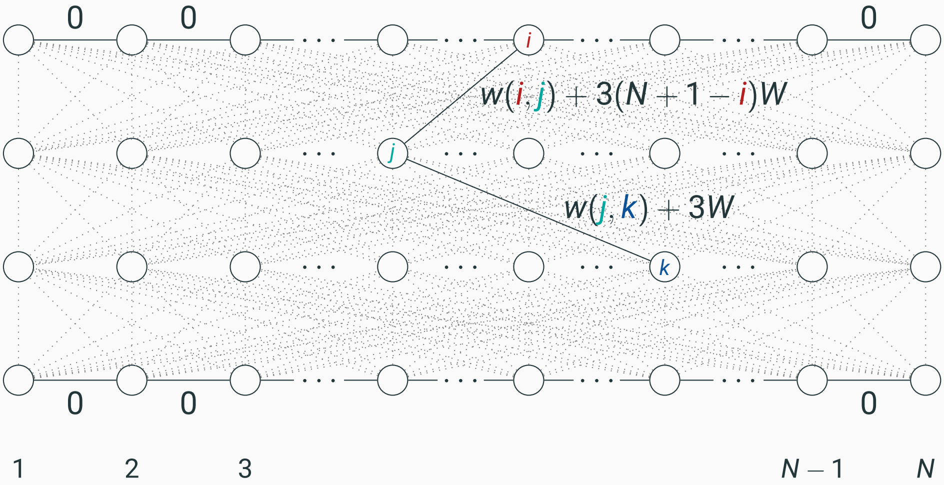

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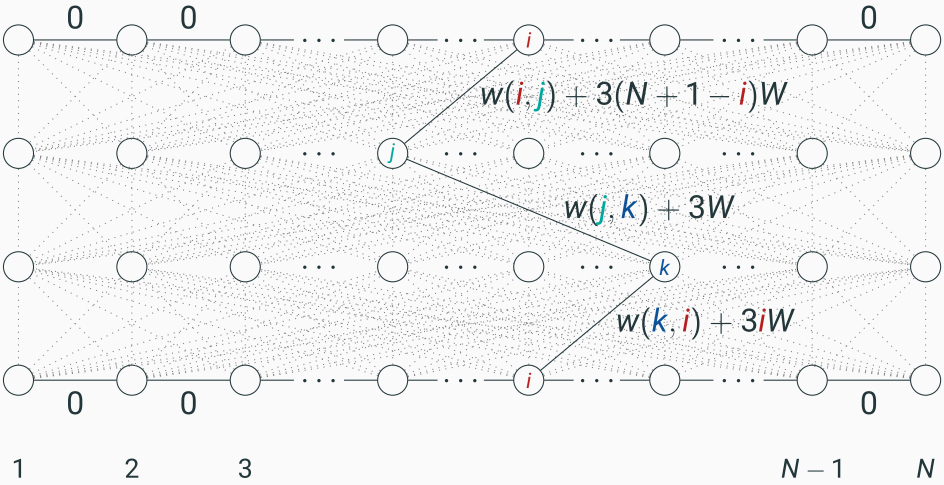

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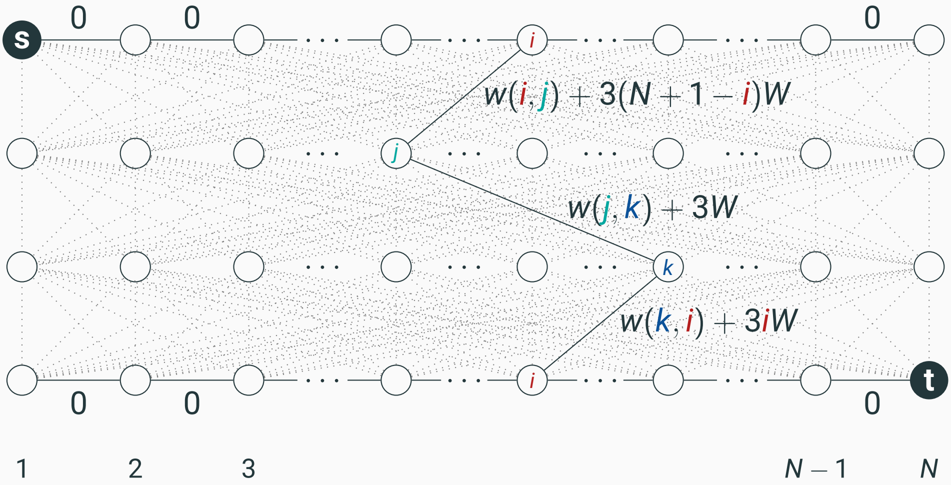

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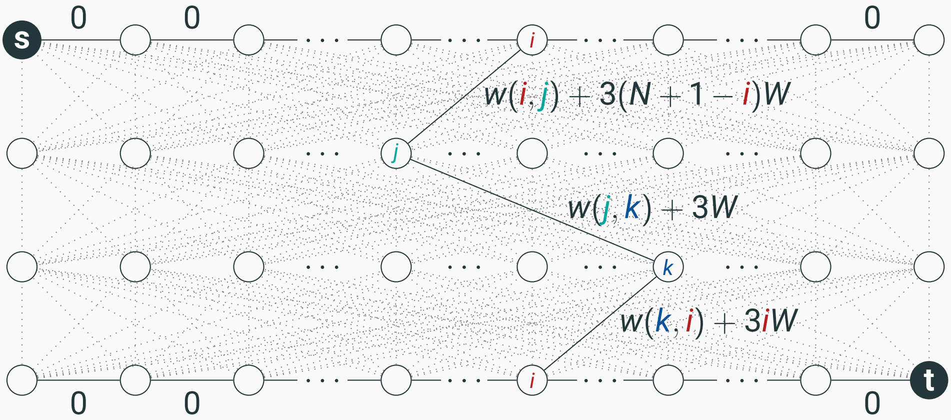

$\exists$ negative triangle $\iff \exists$ s-t path with $\leqslant N+2$ hops of length $< 3(N+2)W$

$\underbrace{\text{negative triangle}}_{i,j,k \,:\, w(i,j)+w(j,k)+w(k,i)<0}$

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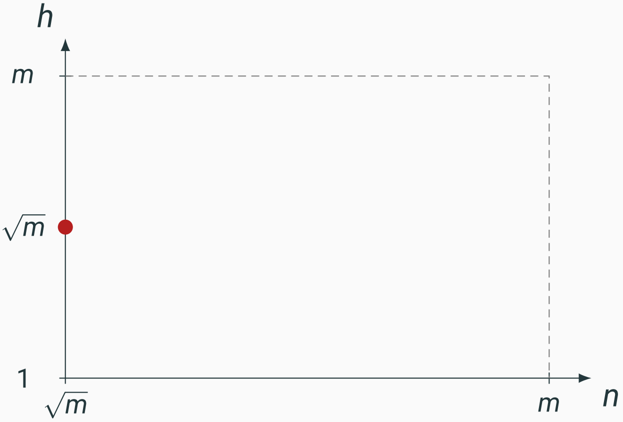
$$O(N^{3-\varepsilon'}) \text{ time} \quad \longleftarrow \quad O(h^{1-\varepsilon} m) \text{ or } O(h m^{1-\varepsilon}) \text{ time}$$

Hardness regime



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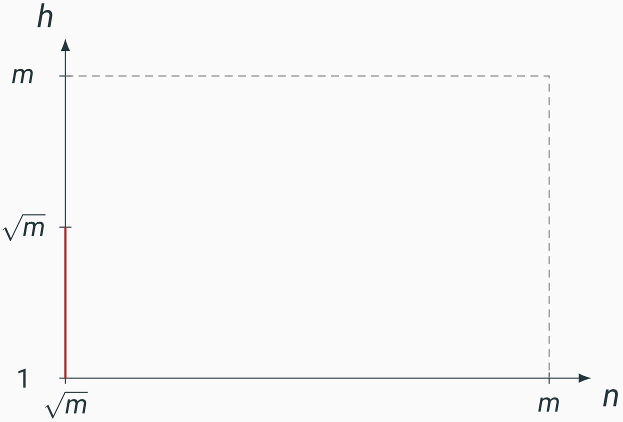

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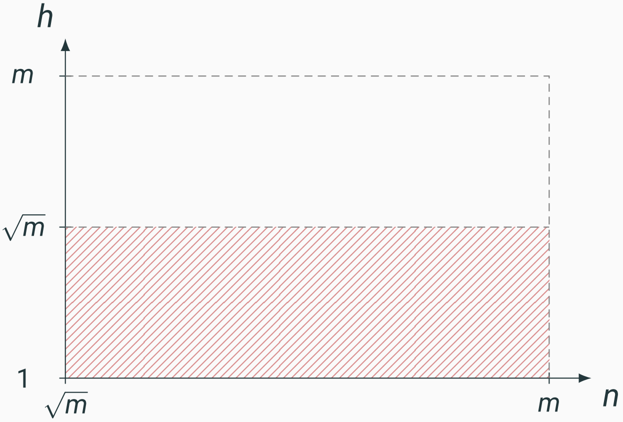

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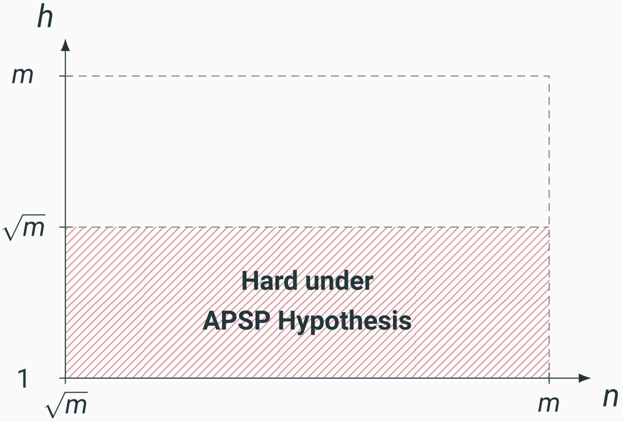

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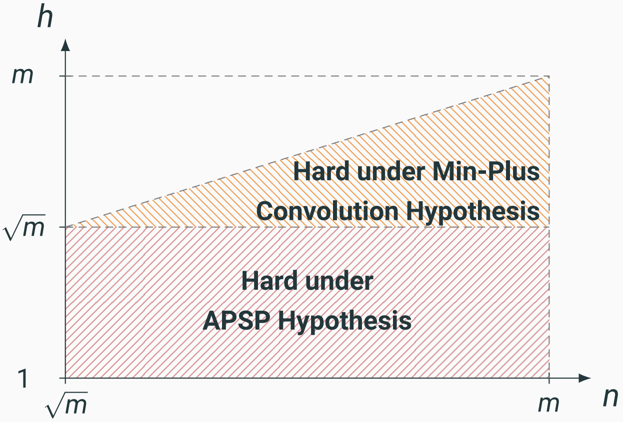

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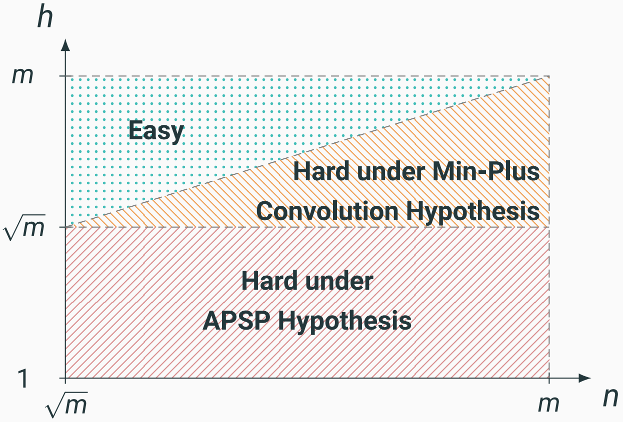

Where else Bellman–Ford might be optimal?

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Open: Negative-weight SSSP faster than $O(nm)$ without $\log W$ factor?

Thank you!