Monochromatic Triangles Intermediate Matrix Products and Convolutions

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matrix multiplication	$C_{ij} = \Sigma_k A_{ik} \times B_{kj}$	$O(n^2)$*
(min,+)-product	$C_{ij} = \min_k A_{ik} + B_{kj}$	$O(n^3)$
(min,max)-product	$C_{ij} = \min_k \max(A_{ik}, B_{kj})$	$O(n^{2.5})$*
Hamming product	$C_{ij} = \Sigma_k \mathbb{1}[A_{ik} = B_{kj}]$	$O(n^{2.5})$*
APSP in unweighted directed graphs		$O(n^{2.5})$*
...		

coincidence?

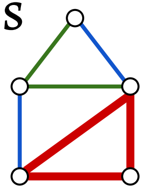
* If ω=2. So far $\omega < 2.373$

our results:
bunch of fine-grained reductions
between intermediate problems

e.g.: APSP in unweighted directed graphs

is no harder than

finding monochromatic triangles
in edge-colored graphs



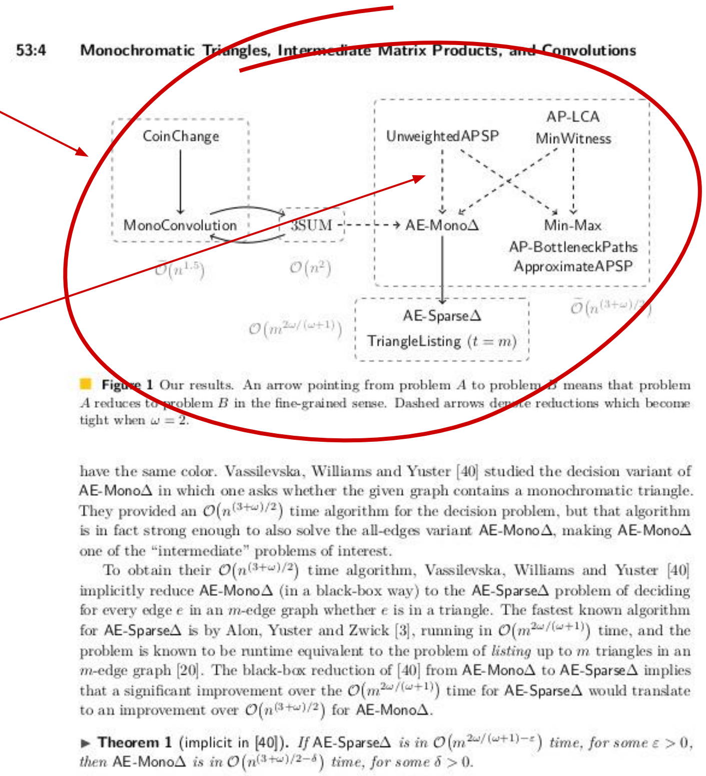

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Figure 1 Our results. An arrow pointing from problem A to problem B means that problem A reduces to problem B in the fine-grained sense. Dashed arrows denote reductions which become tight when $\omega = 2$.

have the same color. Vassilevska, Williams and Yuster [40] studied the decision variant of AE-MonoΔ in which one asks whether the given graph contains a monochromatic triangle. They provided an $\mathcal{O}(n^{(3+\omega)/2})$ time algorithm for the decision problem, but that algorithm is in fact strong enough to also solve the all-edges variant AE-MonoΔ, making AE-MonoΔ one of the "intermediate" problems of interest.

To obtain their $\mathcal{O}(n^{(3+\omega)/2})$ time algorithm, Vassilevska, Williams and Yuster [40] implicitly reduce AE-MonoΔ (in a black-box way) to the AE-SparseΔ problem of deciding for every edge e in an m-edge graph whether e is in a triangle. The fastest known algorithm for AE-SparseΔ is by Alon, Yuster and Zwick [3], running in $\mathcal{O}(m^{2\omega/(\omega+1)})$ time, and the problem is known to be runtime equivalent to the problem of *listing* up to m triangles in an m-edge graph [20]. The black-box reduction of [40] from AE-MonoΔ to AE-SparseΔ implies that a significant improvement over the $\mathcal{O}(m^{2\omega/(\omega+1)})$ time for AE-SparseΔ would translate to an improvement over $\mathcal{O}(n^{(3+\omega)/2})$ for AE-MonoΔ.

▶ **Theorem 1** (implicit in [40]). *If* AE-SparseΔ *is in* $\mathcal{O}(m^{2\omega/(\omega+1)-\varepsilon})$ *time, for some* $\varepsilon > 0$, *then* AE-MonoΔ *is in* $\mathcal{O}(n^{(3+\omega)/2-\delta})$ *time, for some* $\delta > 0$.

Monochromatic Convolution

Input

Output

(i.e. 3SUM-Conjecture fails)

$O(n^{1.5-\varepsilon'})$ time iff 3SUM in $O(n^{2-\varepsilon''})$

first fine-grained equivalence between problems of different complexity