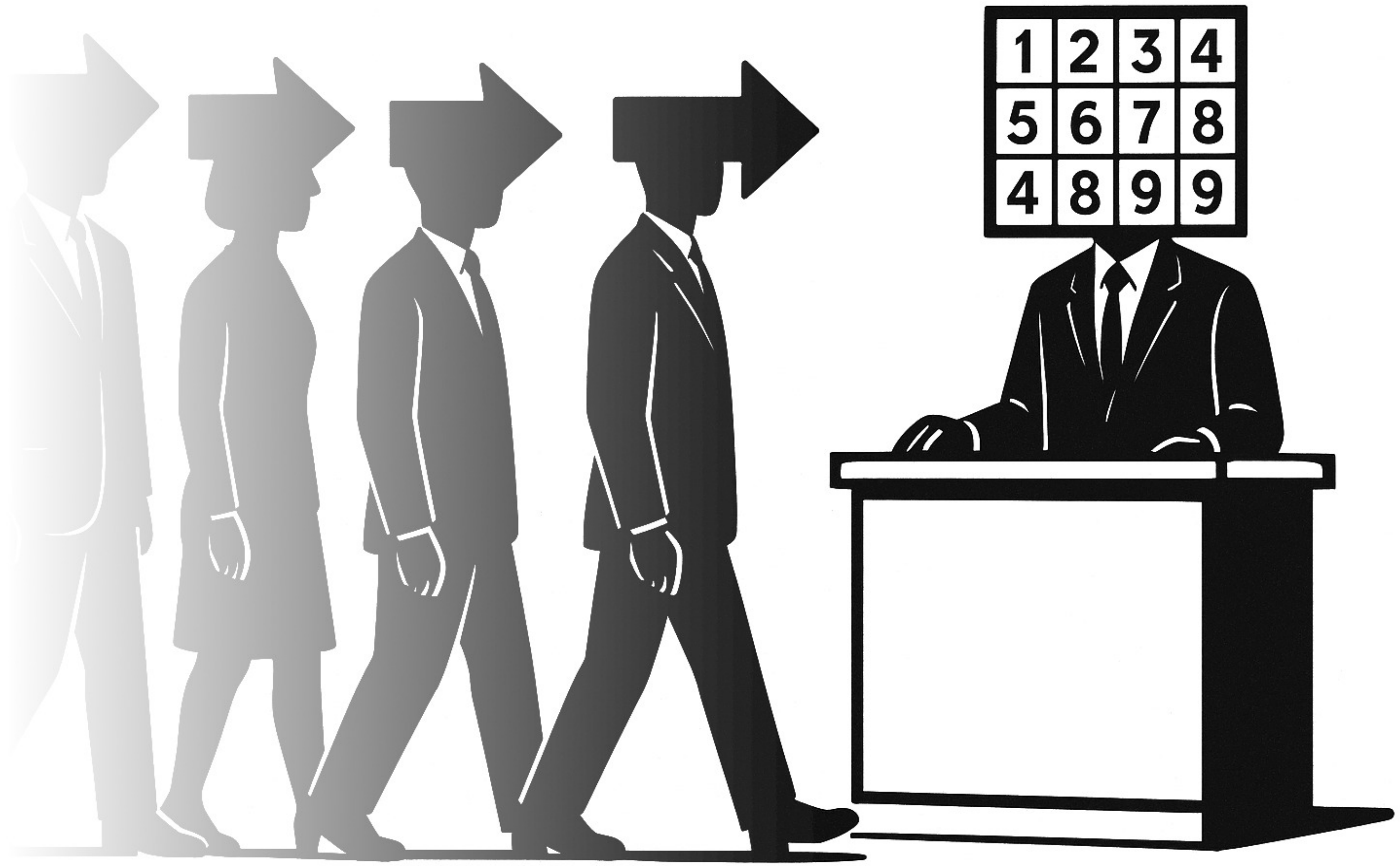
Online Matrix-vector multiplication

Non-Boolean OMv

ONE MORE REASON TO BELIEVE LOWER BOUNDS FOR DYNAMIC PROBLEMS



Bingbing Hu

Adam Polak

UC San Diego Bocconi

Dynamic algorithms

= data structures, but for fancier queries

Operations:

- **updates**
  e.g., add edge, delete edge

- **queries**
  e.g., find a path from source to v,
  is the graph strongly connected?

Dynamic algorithms

= data structures, but for fancier queries

Operations:

- **updates**
  e.g., add edge, delete edge

- **queries**
  e.g., find a path from source to v,
  is the graph strongly connected?

recompute from scratch

sublinear time update/query

polylog time update/query

Dynamic algorithms

= data structures, but for fancier queries

Operations:

- **updates**
  e.g., add edge, delete edge
- **queries**
  e.g., find a path from source to v,
  is the graph strongly connected?

Fine-grained **hypotheses**

3SUM

APSP

SETH

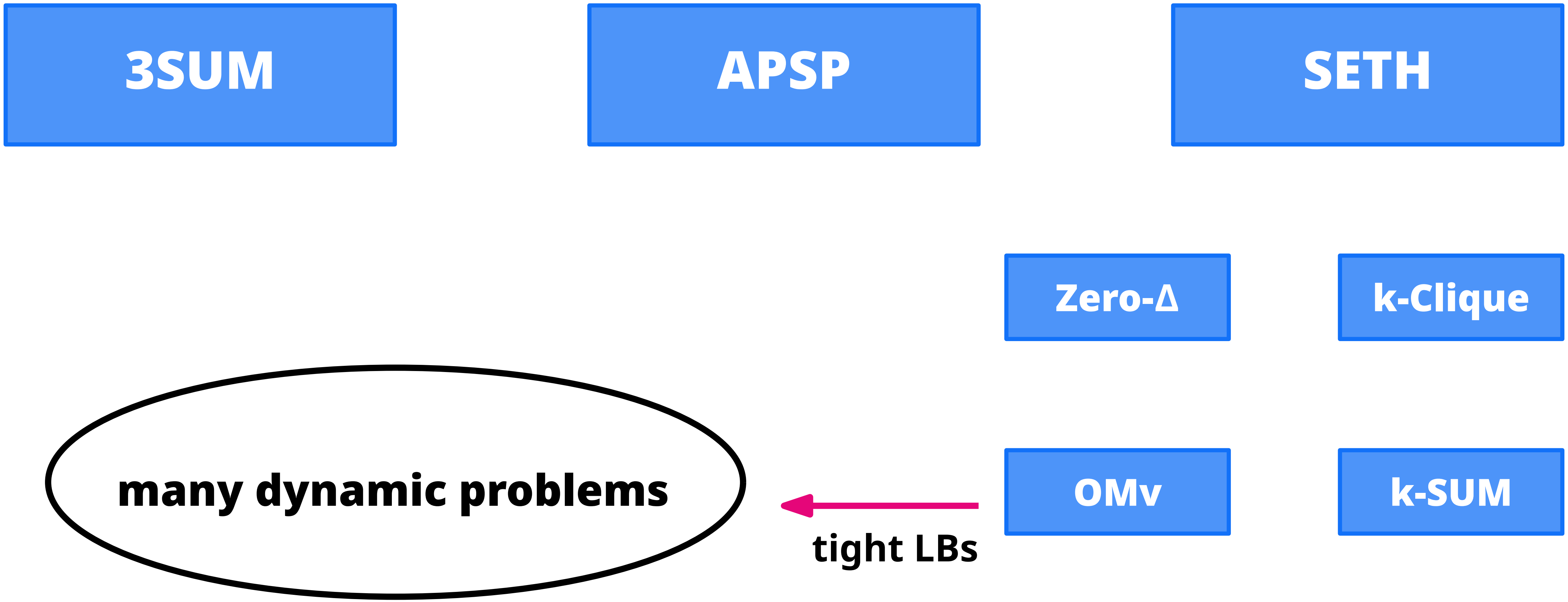
Zero-Δ

k-Clique

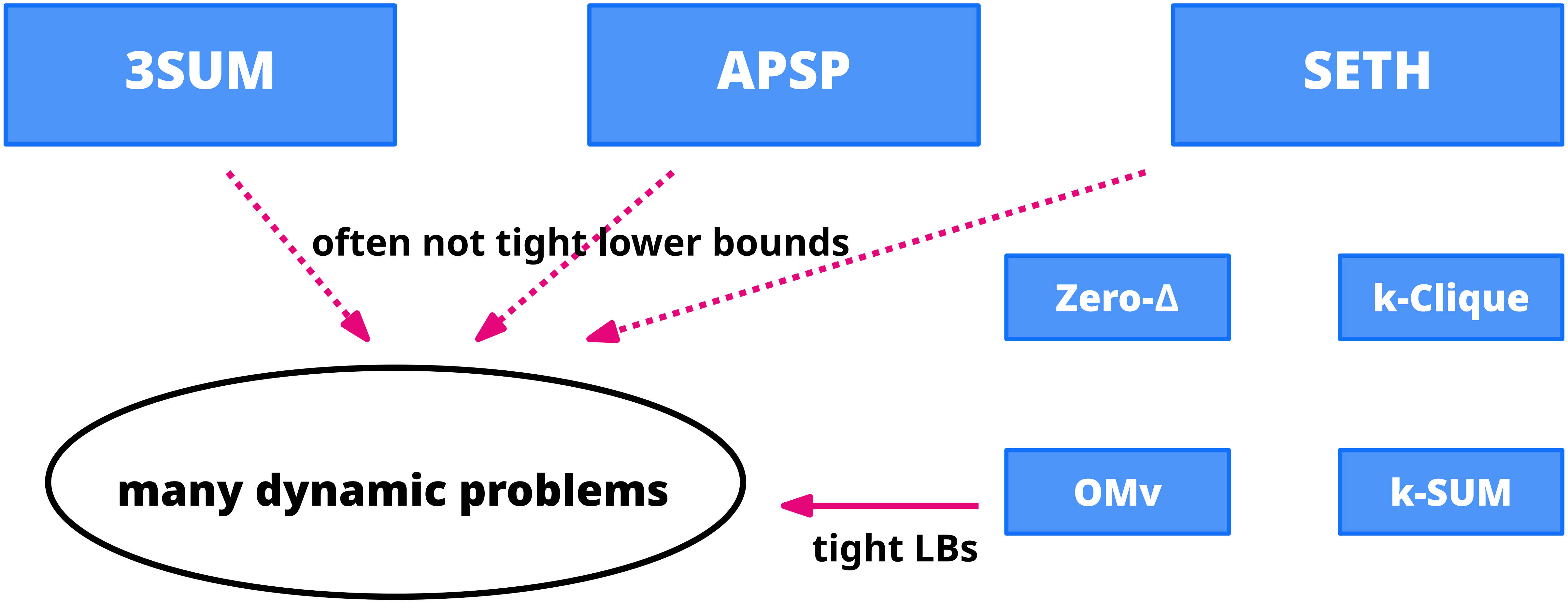
OMv

k-SUM

Fine-grained **hypotheses**



Fine-grained hypotheses



Can we have tight lower bounds for dynamic problems based on a hypothesis that is more believable than OMv?

(Boolean) Online Matrix-vector multiplication (OMv)

[Henzinger–Krinninger–Nanongkai–Saranurak, STOC'15]

Input:

Boolean $n \times n$ matrix $\boldsymbol{A}$,

and n Boolean vectors $\boldsymbol{v_1, v_2, \ldots, v_n}$ given **online**

Output:

Boolean products $\boldsymbol{Mv_1, Mv_2, \ldots, Mv_n}$

must output $\boldsymbol{Mv_i}$ **before being able to see** $\boldsymbol{v_{i+1}}$

(Boolean) Online Matrix-vector multiplication (OMv)

[Henzinger–Krinninger–Nanongkai–Saranurak, STOC'15]

Input:
Boolean $n \times n$ matrix $\boldsymbol{A}$,
and n Boolean vectors $\boldsymbol{v_1, v_2, \ldots, v_n}$ given **online**

Output:
Boolean products $\boldsymbol{Mv_1, Mv_2, \ldots, Mv_n}$
must output $\boldsymbol{Mv_i}$ **before being able to see** $\boldsymbol{v_{i+1}}$

OMv Hypothesis: No $\boldsymbol{O(n^{3-\epsilon})}$ time algorithm for OMv

Various matrix products (static)

Easy

$O(n^{\omega}) < O(n^{2.372})$

- Integer product
- Boolean product

Various matrix products (static)

Easy

$O(n^{\omega}) < O(n^{2.372})$

- Integer product
- Boolean product

Hard

$n^{3-o(1)}$

- Min-plus product

Various matrix products (static)

Easy

$O(n^{\omega}) < O(n^{2.372})$

- Integer product
- Boolean product

Intermediate

$O(n^{\frac{3+\omega}{2}}) < O(n^{2.686})$

- Min-witness product
- Min-max product
- Dominance product
- Equality product
- Bounded monotone min-plus product

Hard

$n^{3-o(1)}$

- Min-plus product

Min-Max Online Matrix-vector multiplication

$$\boxed{(M \ovee v)[i] := \min_j \max(M[i][j], v[j])}$$

Input:
Integer $n \times n$ matrix $\boldsymbol{A}$,
and n integer vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}$ given **online**

Output:
Min-Max products $\boldsymbol{M} \ovee \boldsymbol{v_1}, \boldsymbol{M} \ovee \boldsymbol{v_2}, \ldots, \boldsymbol{M} \ovee \boldsymbol{v_n}$
must output $\boldsymbol{M} \ovee \boldsymbol{v_i}$ **before being able to see** $\boldsymbol{v_{i+1}}$

Min-Max Online Matrix-vector multiplication

$$\boxed{(M \circledvee v)[i] := \min_j \max(M[i][j], v[j])}$$

Input:
Integer $n \times n$ matrix $\boldsymbol{A}$,
and n integer vectors $\boldsymbol{v_1, v_2, \ldots, v_n}$ given **online**

Output:
Min-Max products $\boldsymbol{M \circledvee v_1, M \circledvee v_2, \ldots, M \circledvee v_n}$
must output $\boldsymbol{M \circledvee v_i}$ **before being able to see** $\boldsymbol{v_{i+1}}$

Min-Max-OMv Hypothesis: No $\boldsymbol{O(n^{3-\epsilon})}$ time algorithm for Min-Max-OMv

Min-Max Online Matrix-vector multiplication

$$\boxed{(M \varovee v)[i] := \min_j \max(M[i][j], v[j])}$$

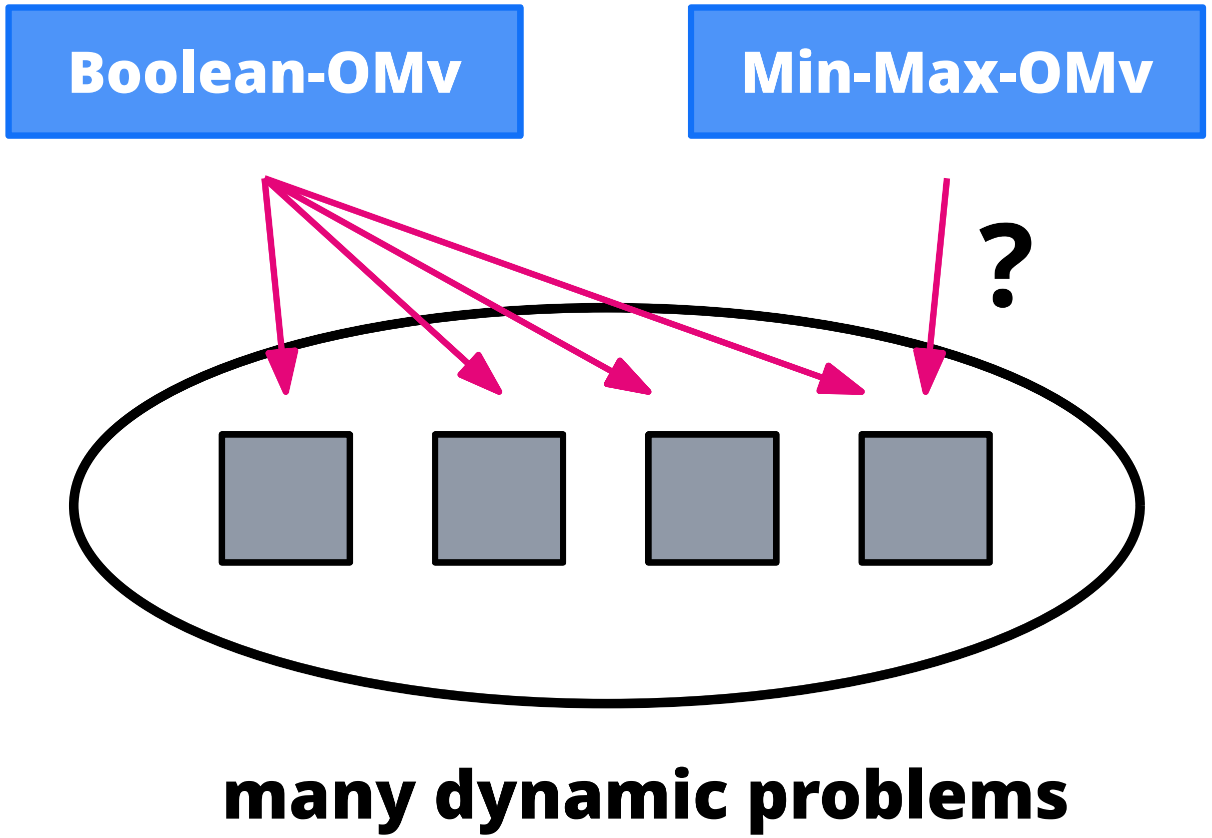
Input:
Integer $n \times n$ matrix $\boldsymbol{A}$,
and n integer vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}$ given **online**

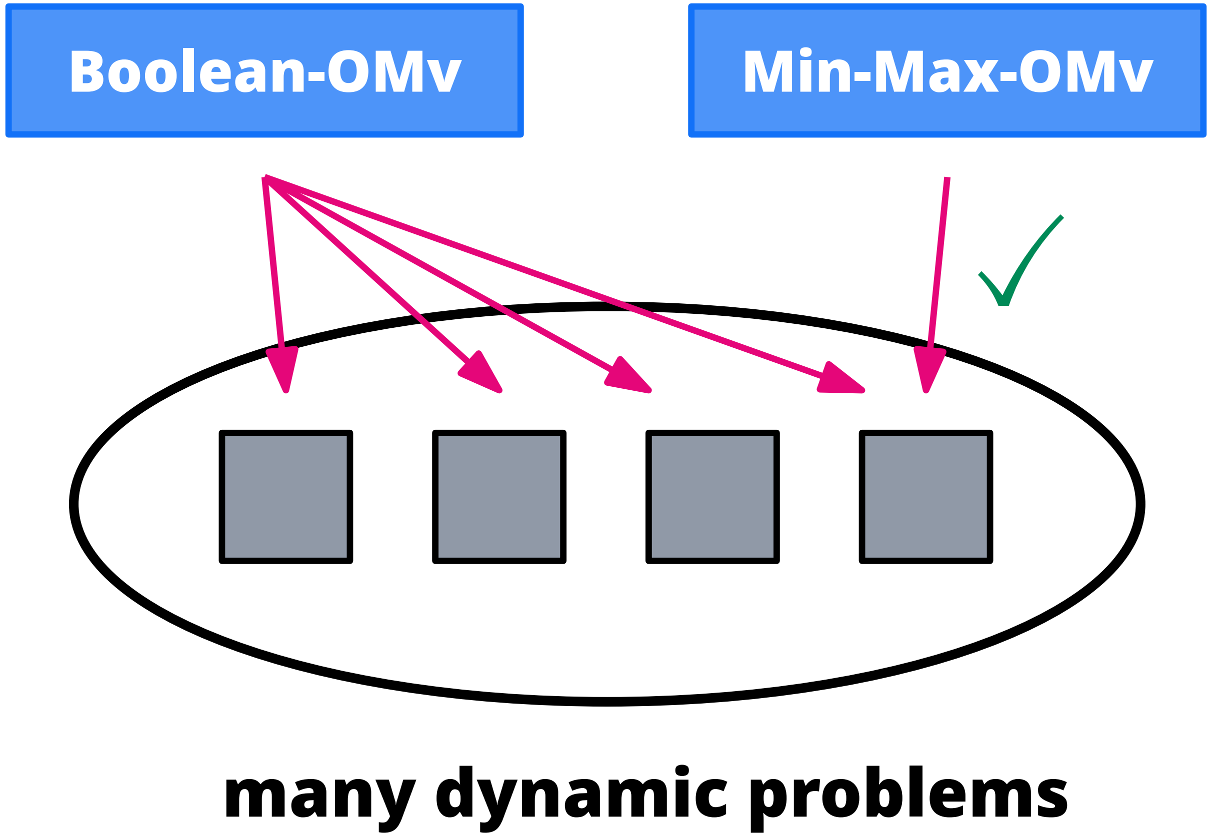
Output:
Min-Max products $\boldsymbol{M \varovee v_1}, \boldsymbol{M \varovee v_2}, \ldots, \boldsymbol{M \varovee v_n}$
must output $\boldsymbol{M \varovee v_i}$ **before being able to see** $\boldsymbol{v_{i+1}}$

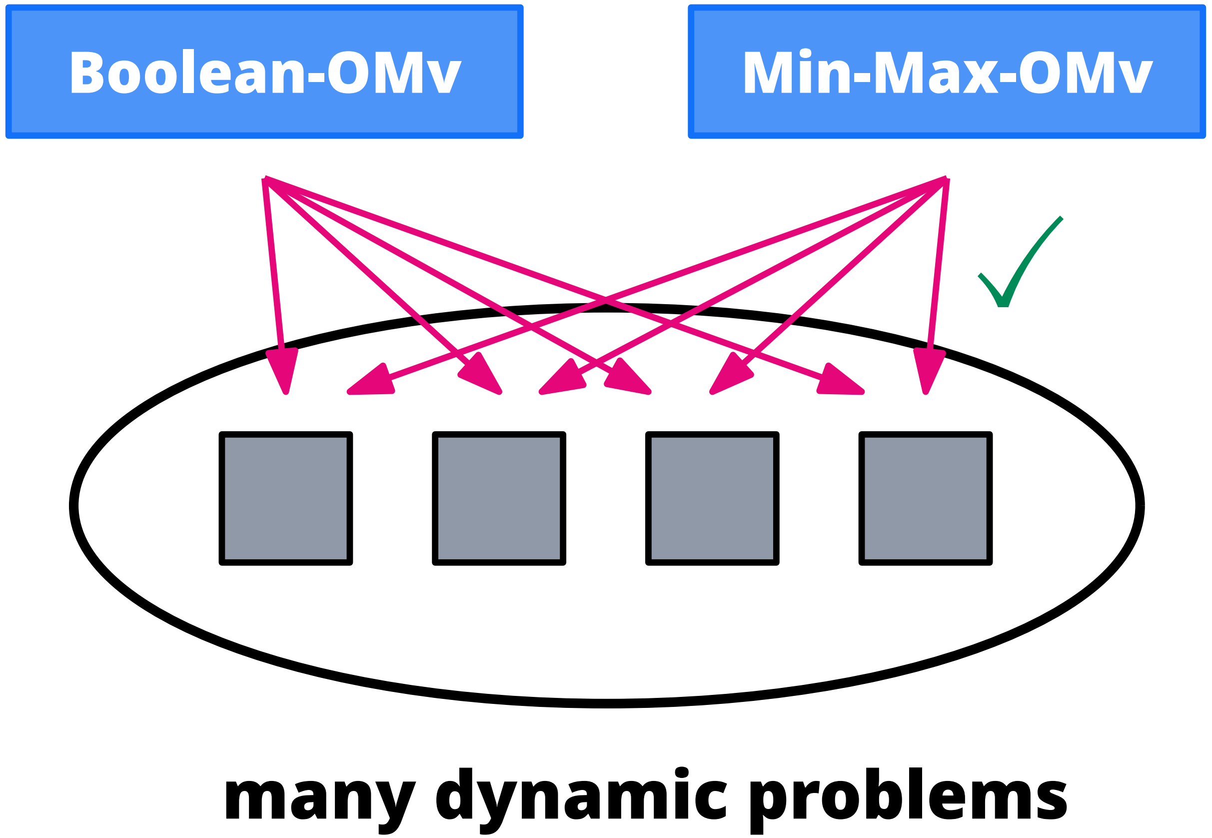
Min-Max-OMv Hypothesis: No $\boldsymbol{O(n^{3-\epsilon})}$ time algorithm for Min-Max-OMv

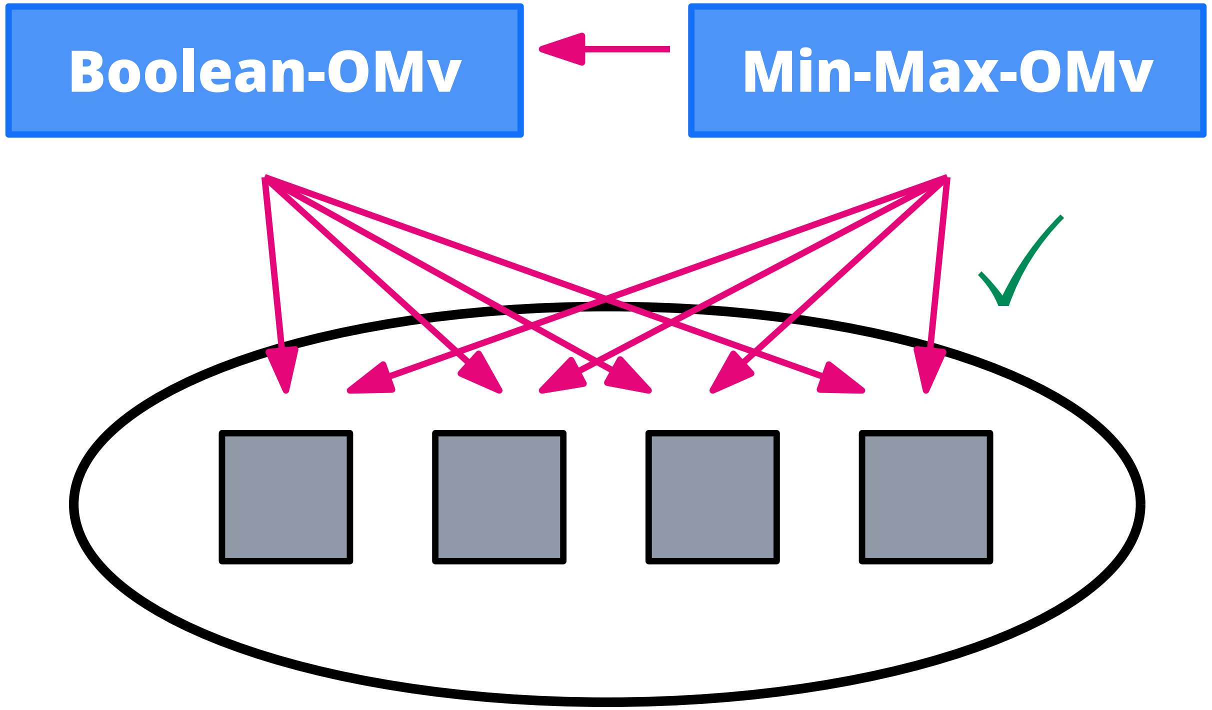
a priori *more believable* than **(Boolean-)OMv Hypothesis**

Can we give tight reductions
from Min-Max-OMv to those dynamic problems
that have known tight reductions from (Boolean-)OMv?

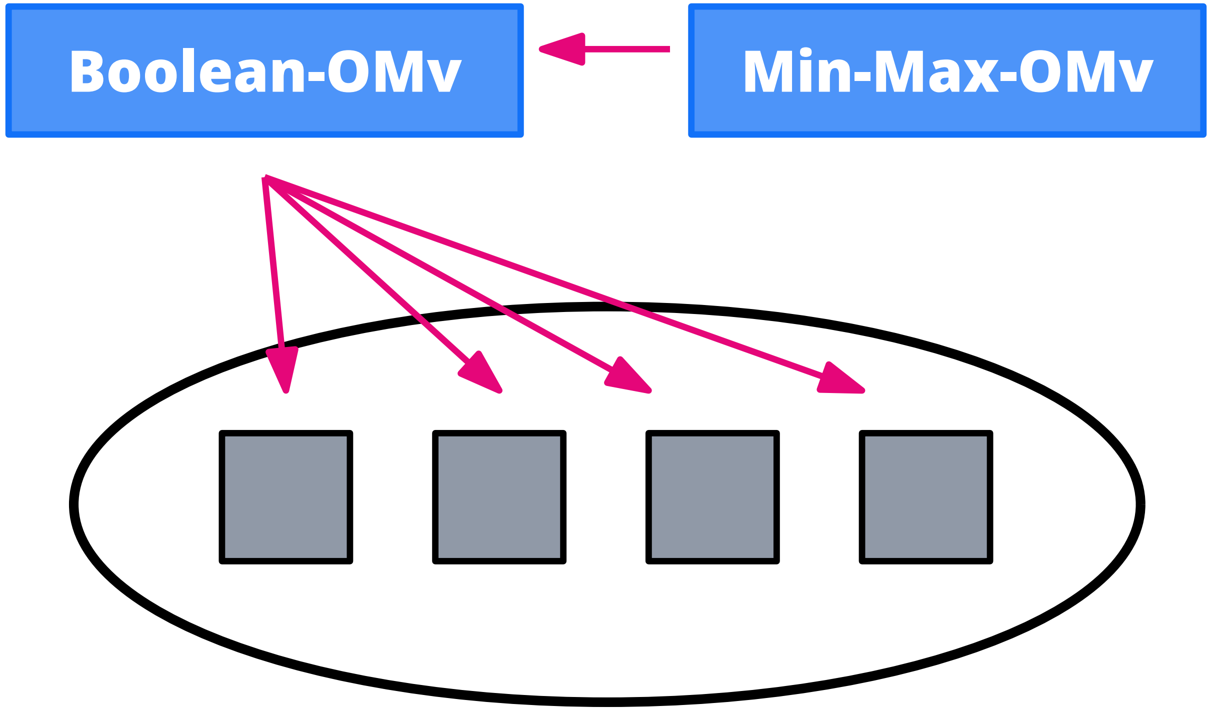

Boolean-OMv
Min-Max-OMv
?
many dynamic problems


Boolean-OMv
Min-Max-OMv
many dynamic problems

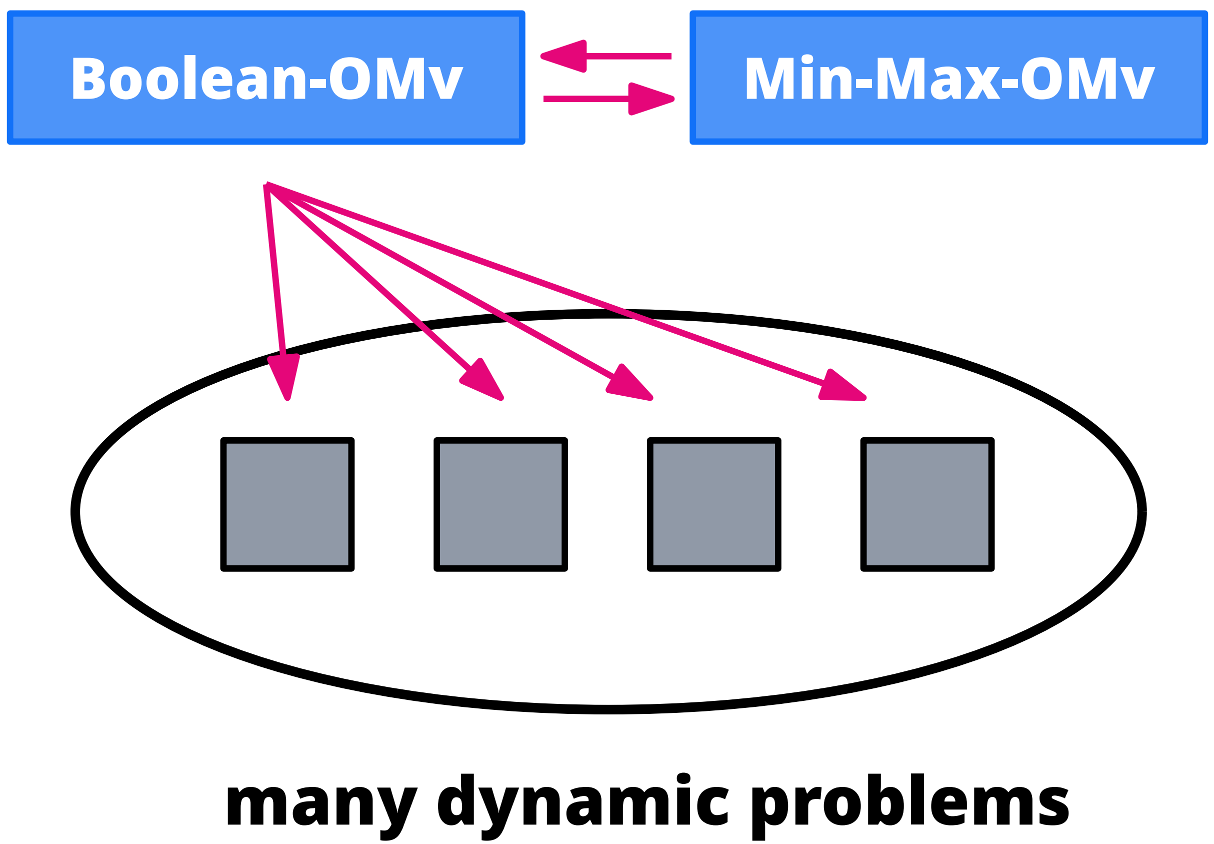

Boolean-OMv
Min-Max-OMv
many dynamic problems

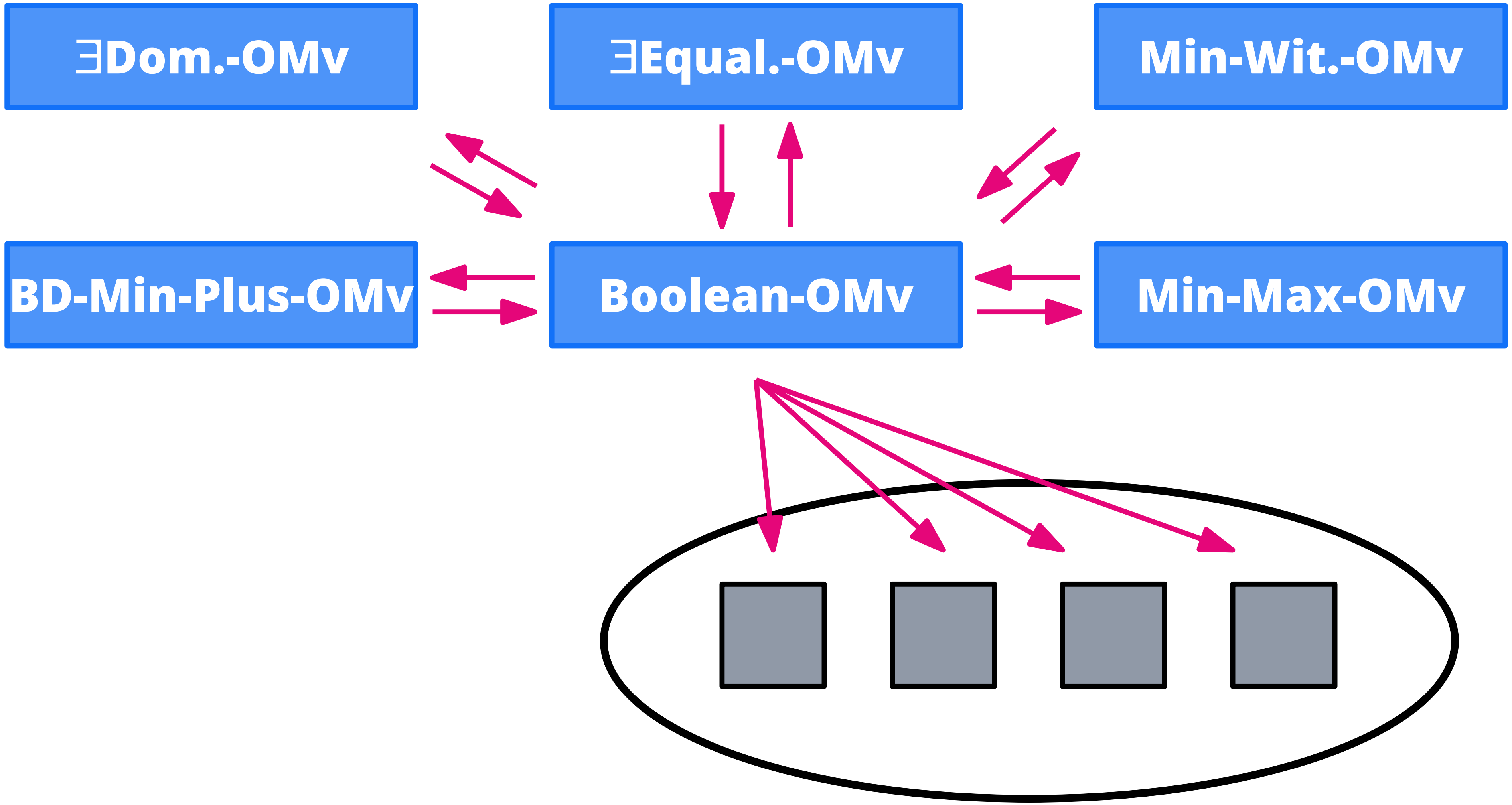


many dynamic problems



many dynamic problems


Boolean-OMv
Min-Max-OMv
many dynamic problems



many dynamic problems

Our **theorem**

These problems **either all** have **truly subcubic** algorithms **or none** of them have:

- Boolean-OMv; $(\exists_k M[i,k] \wedge v[k])$
- ∃Equality-OMv; $(\exists_k M[i,k] = v[k])$
- ∃Dominance-OMv; $(\exists_k M[i,k] \leqslant v[k])$
- Min-Witness-OMv; $(\min \{k \mid M[i,k] \wedge v[k]\})$
- Min-Max-OMv; $(\min_k \max\{M[i,k], v[k]\})$
- Bounded Monotone Min-Plus-OMv. $(\min_k M[i,k] + v[k])$

Our theorem

These problems **either all** have **truly subcubic** algorithms **or none** of them have:

- Boolean-OMv; $(\exists_k\, M[i,k] \wedge v[k])$
- $\exists$Equality-OMv; $(\exists_k\, M[i,k] = v[k])$
- $\exists$Dominance-OMv; $(\exists_k\, M[i,k] \leqslant v[k])$
- Min-Witness-OMv; $(\min\{k \mid M[i,k] \wedge v[k]\})$
- Min-Max-OMv; $(\min_k \max\{M[i,k], v[k]\})$
- Bounded Monotone Min-Plus-OMv. $(\min_k M[i,k] + v[k])$

Surprise? Not in hindsight.

Known **static** algorithms in time $O\left(n^{\frac{3+\omega}{2}}\right)$

i.e. $O(n^{f(\omega)})$ s.t. $x < 3 \implies f(x) < 3$

Our theorem

These problems **either all** have **truly subcubic** algorithms **or none** of them have:

- Boolean-OMv; $(\exists_k\, M[i,k] \wedge v[k])$
- ∃Equality-OMv; $(\exists_k\, M[i,k] = v[k])$
- ∃Dominance-OMv; $(\exists_k\, M[i,k] \leqslant v[k])$
- Min-Witness-OMv; $(\min\{k \mid M[i,k] \wedge v[k]\})$
- Min-Max-OMv; $(\min_k \max\{M[i,k], v[k]\})$
- Bounded Monotone Min-Plus-OMv $(\min_k M[i,k] + v[k])$

Surprise? Not in hindsight.

Known **static** algorithms
in time $O\left(n^{\frac{3+\omega}{2}}\right)$

i.e. $O(n^{f(\omega)})$ s.t.
$x < 3 \implies f(x) < 3$

Remark: If (static) BMM has a subcubic *combinatorial* algorithm,
then all these variants have such algorithms as well.

Our **theorem**

Open problem: Add a **counting** variant to the list.

These problems **either all** have **truly subcubic** algorithms **or none** of them have:

- Boolean-OMv; $(\exists_k\, M[i,k] \wedge v[k])$
- ∃Equality-OMv; $(\exists_k\, M[i,k] = v[k])$
- ∃Dominance-OMv; $(\exists_k\, M[i,k] \leqslant v[k])$
- Min-Witness-OMv; $(\min\{k \mid M[i,k] \wedge v[k]\})$
- Min-Max-OMv; $(\min_k \max\{M[i,k], v[k]\})$
- Bounded Monotone Min-Plus-OMv $(\min_k M[i,k] + v[k])$

Surprise? Not in hindsight.

Known **static** algorithms in time $O\left(n^{\frac{3+\omega}{2}}\right)$

i.e. $O(n^{f(\omega)})$ s.t. $x < 3 \implies f(x) < 3$

Remark: If (static) BMM has a subcubic *combinatorial* algorithm, then all these variants have such algorithms as well.

THANK YOU!