Equivalences Between Triangle and Range Query Problems

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EdgeTriangleCounting

In: an m-edge graph G

Out: for each edge $e \in G$ the number of triangles containing e

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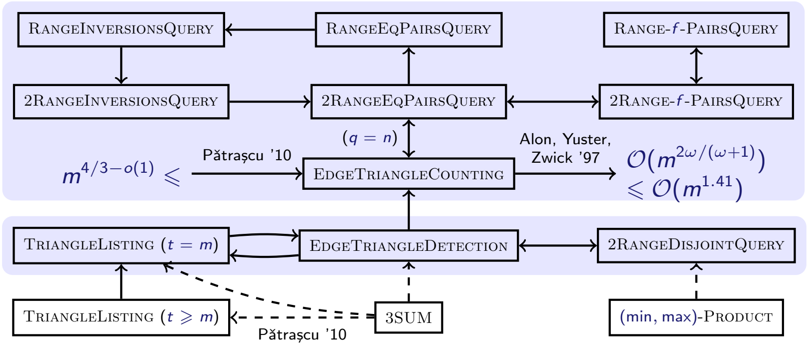
EdgeTriangleCounting $\leftrightarrow$ RangeInversionsQuery

If EdgeTriangleCounting can be solved in $T(m)$ time, then RangeInversionsQuery can be solved in $\widetilde{\mathcal{O}}(T(n+q))$ time.

If RangeInversionsQuery can be solved in $T(n+q)$ time, then EdgeTriangleCounting can be solved in $\widetilde{\mathcal{O}}(T(m))$ time.


RangeInversionsQuery
RangeEqPairsQuery
Range-f-PairsQuery
2RangeInversionsQuery
2RangeEqPairsQuery
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$(q = n)$
Alon, Yuster, Zwick '97
Pătrașcu '10
$m^{4/3-o(1)} \leqslant$
EdgeTriangleCounting
$\mathcal{O}(m^{2\omega/(\omega+1)}) \leqslant \mathcal{O}(m^{1.41})$
TriangleListing $(t = m)$
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2RangeDisjointQuery
TriangleListing $(t \geqslant m)$
3SUM
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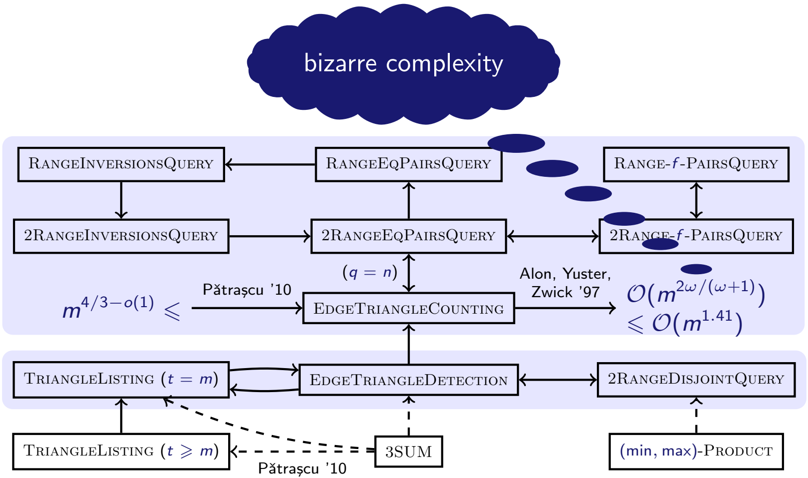
(min, max)-Product



bizarre complexity
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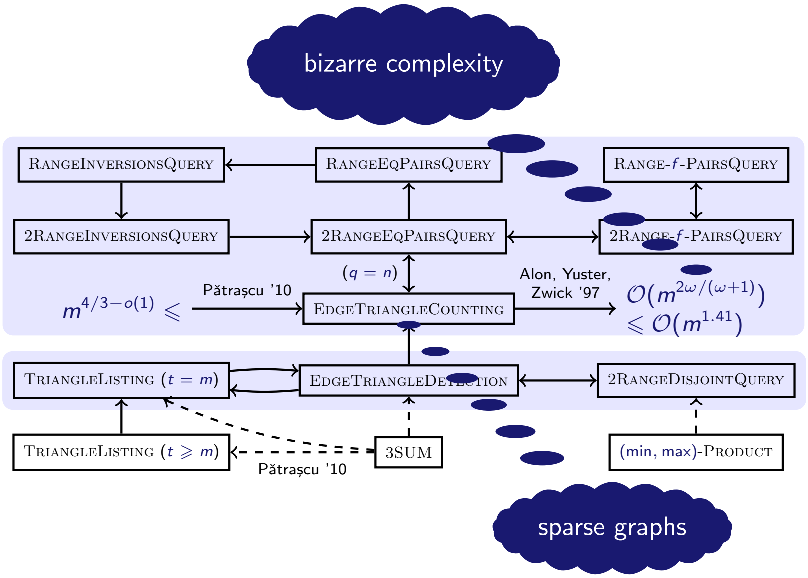
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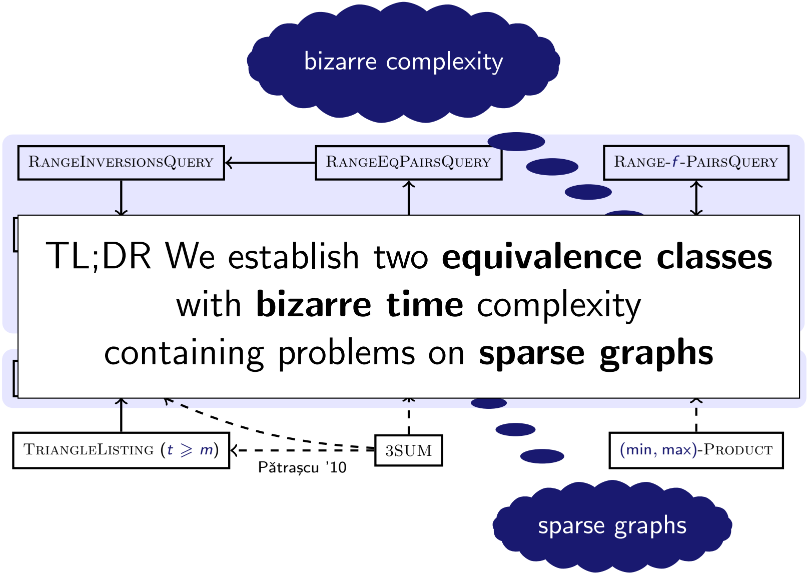
sparse graphs



TL;DR We establish two **equivalence classes** with **bizarre time** complexity containing problems on **sparse graphs**

RangeInversionsQuery

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- Is $\widetilde{\mathcal{O}}(n^{1.49})$ possible?

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2RangeInversionsQuery

In: an integer array $A[1..n]$, and queries $([l_i', r_i'], [l_i'', r_i''])_{i \in [q]}$ $(r_i' < l_i'')$

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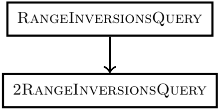
RangeInversionsQuery
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RangeInversionsQuery → 2RangeInversionsQuery

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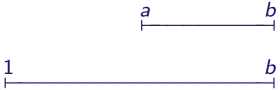

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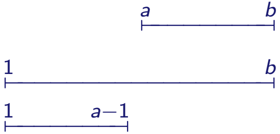

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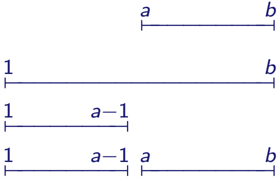

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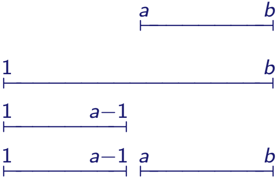

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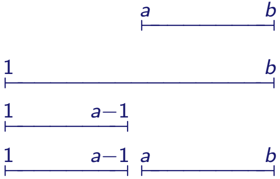

RangeInversionsQuery

preprocess $\operatorname{inv}\big([1, i]\big)$ for all i's, in $\mathcal{O}(n \log n)$

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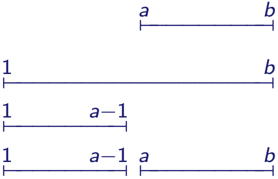

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$\mathrm{inv}\left([a,b]\right) =$ — RangeInversionsQuery

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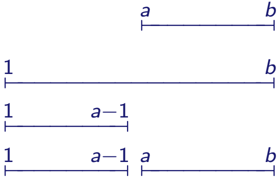
$-\,\mathrm{inv}\left([1,a-1]\right)$

$-\,\mathrm{inv}\left([1,a-1],[a,b]\right)$



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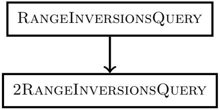
Total time: $\mathcal{O}(n \log n) + T(n,q) = \widetilde{\mathcal{O}}(T(n,q))$ □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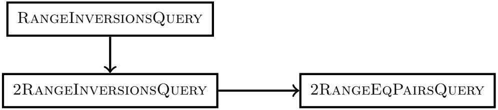

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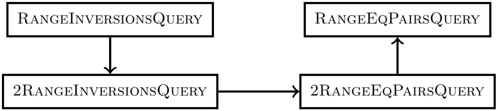

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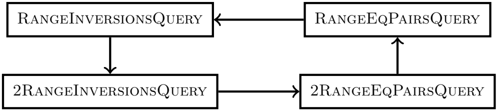

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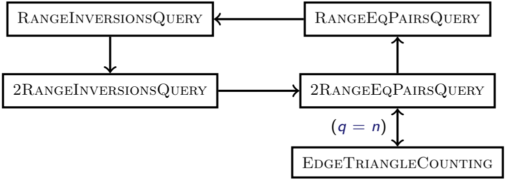
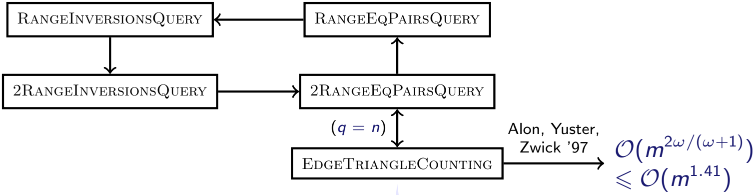


In: a (sparse) graph G
Out: for every edge $e \in G$
$\#$ of triangles containing e



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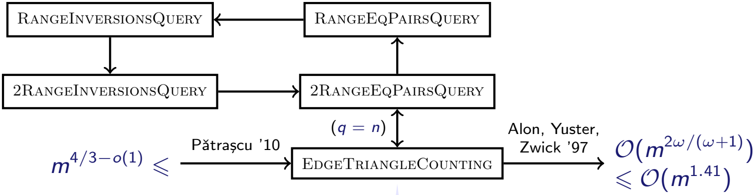

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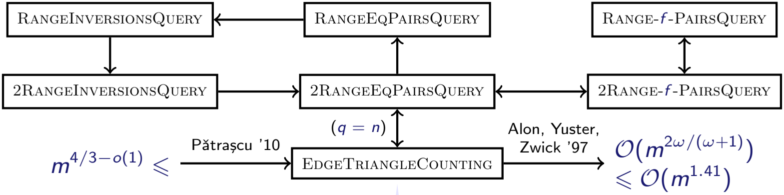
$\sum_{l_i \leqslant x < y \leqslant r_i} f(A[x], A[y]) = ?$



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$f : \mathbb{Z}^2 \to \mathbb{Z}$, **piecewise polynomial non-axis-orthogonal**

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Def. f is piecewise polynomial [LUW'19]

- $f(x, y) = \sum_k P_k(x, y) \cdot \mathbb{1}\left[A_k x + B_k y + C_k > 0\right]$

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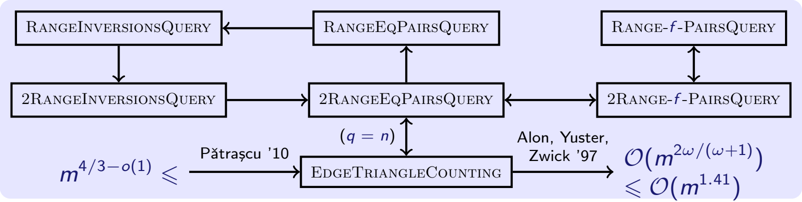
E.g. $=$, $\leqslant$, min, max, ℓ_1, ℓ_{2p+1}, $\mathbb{1}[x - y > \delta]$, $\max(0, x - y)$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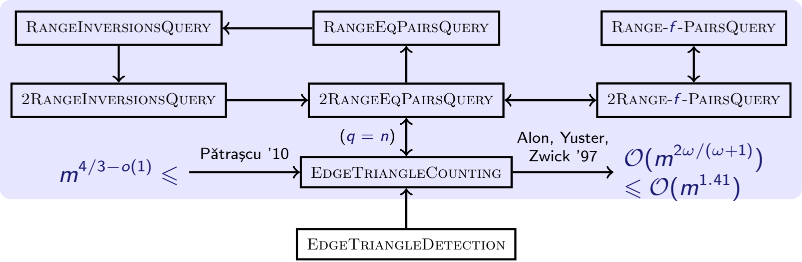
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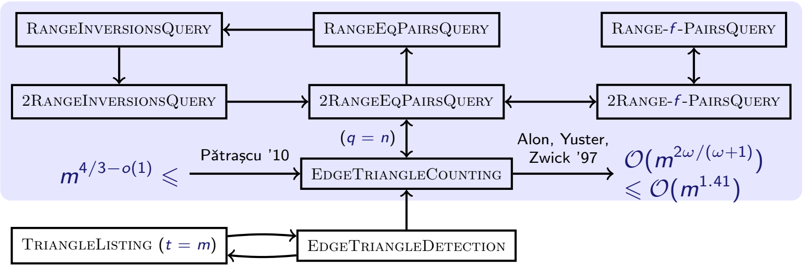
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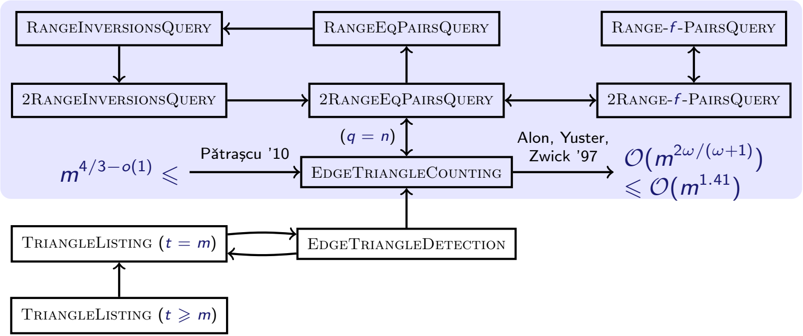

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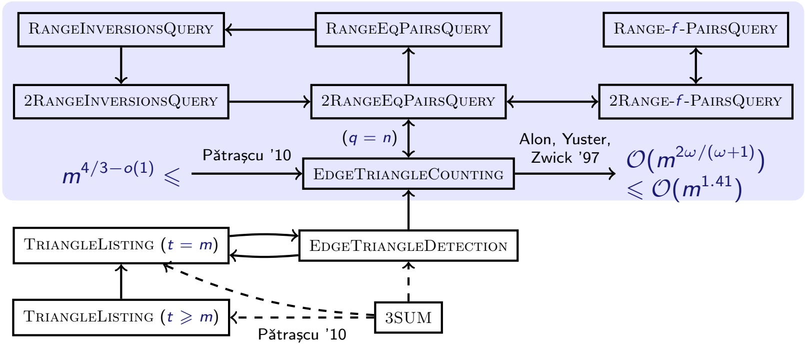

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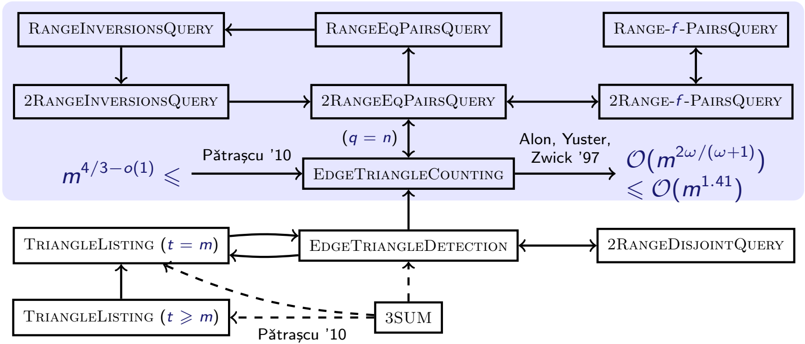
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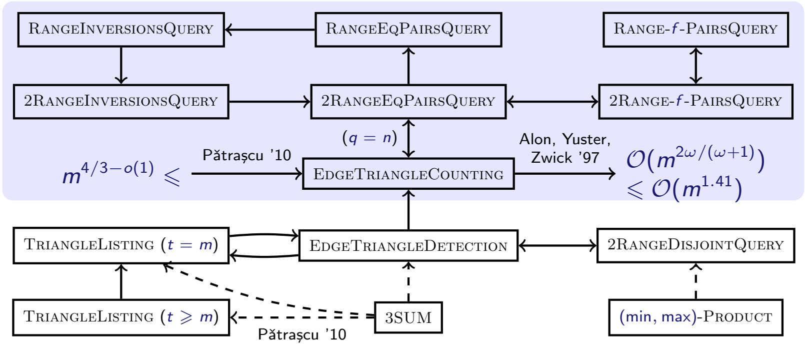
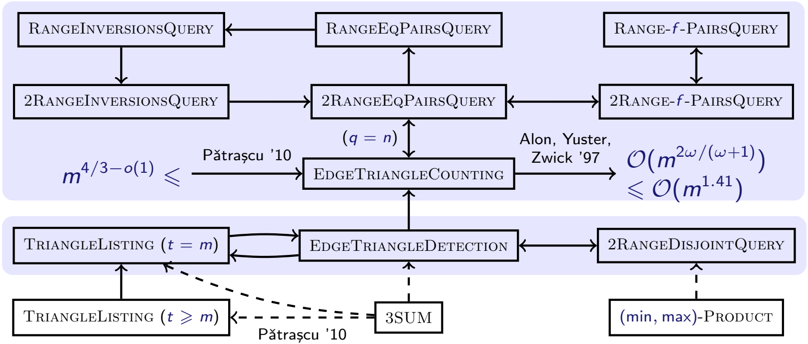
Pătrașcu '10



RangeInversionsQuery
RangeEqPairsQuery
Range-f-PairsQuery
2RangeInversionsQuery
2RangeEqPairsQuery
2Range-f-PairsQuery
$(q = n)$
Alon, Yuster, Zwick '97
Pătrașcu '10
$m^{4/3-o(1)} \leqslant$
EdgeTriangleCounting
$\mathcal{O}(m^{2\omega/(\omega+1)}) \leqslant \mathcal{O}(m^{1.41})$
TriangleListing $(t = m)$
EdgeTriangleDetection
2RangeDisjointQuery
TriangleListing $(t \geqslant m)$
3SUM
(min, max)-Product
Pătrașcu '10


RangeInversionsQuery
RangeEqPairsQuery
Range-f-PairsQuery
2RangeInversionsQuery
2RangeEqPairsQuery
2Range-f-PairsQuery
(q = n)
Alon, Yuster, Zwick '97
Pătrașcu '10
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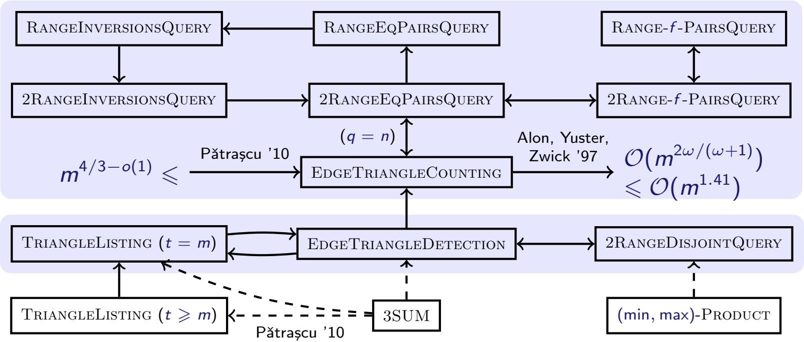
TriangleListing (t = m)
EdgeTriangleDetection
2RangeDisjointQuery
TriangleListing (t ⩾ m)
3SUM
(min, max)-Product
Pătrașcu '10



RangeInversionsQuery
RangeEqPairsQuery
Range-f-PairsQuery
2RangeInversionsQuery
2RangeEqPairsQuery
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$(q = n)$
Pătrașcu '10
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3SUM
Pătrașcu '10
(min, max)-Product

Thank you!