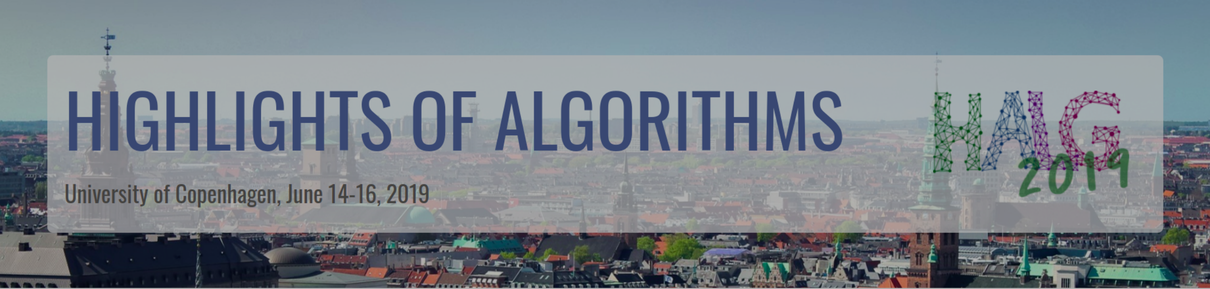
KNAPSACK AND SUBSET SUM WITH SMALL ITEMS

Adam Polak, Lars Rohwedder, Karol Węgrzycki

EPFL · Maastricht University · Universität des Saarlandes · max planck institut informatik



HIGHLIGHTS OF ALGORITHMS

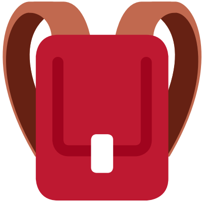
HALG 2019

University of Copenhagen, June 14-16, 2019

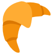
Saturday, June 15

9:00 - 9:55	**Thomas Vidick** — Survey on quantum program checking and quantum multiprover interactive proof systems	**[abstract]**
10:00 - 10:25	**Cliff Stein** — A fast algorithm for knapsack via convolution and prediction	**[abstract]**
10:30 - 11:00	**Coffee Break**	
11:00 - 11:25	**Sebastien Bubeck** — k-server via multiscale entropic regularization	**[+]**
11:30 - 11:55	**Nima Anari** — Planar Graph Perfect Matching is in NC	**[abstract]**
12:00 - 12:25	**Greg Bodwin** — On the Structure of Unique Shortest Paths in Graphs	**[abstract]**

KNAPSACK



$$\underbrace{t = 1500\text{g}}_{\text{max capacity}}$$

$\overset{?}{\longleftarrow}$		1000g	\$300
$\overset{?}{\longleftarrow}$		420g	\$50
$\overset{?}{\longleftarrow}$		85g	\$2
$\overset{?}{\longleftarrow}$		415g	\$49

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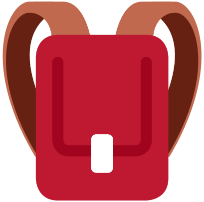

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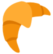
✓ ←		85g	\$2
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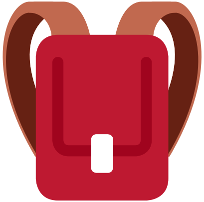

$t = 1500\text{g}$ (max capacity)

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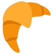
✓ ←		85g	$2
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$O(nt)$

[Bellman, 1957]

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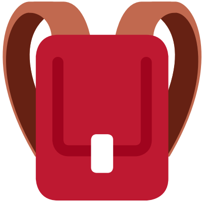

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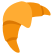
✓ ←		85g	$2
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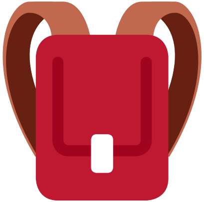

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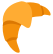
✓ ←		85g	\$2
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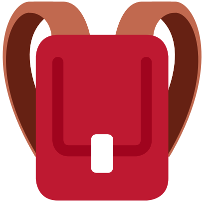

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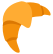
✓		85g	$2
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[Bateni et al., 2018]

[Axiotis-Tzamos, 2019]

† Essentially optimal, unless Min-Plus Convolution Conjecture fails

[Cygan et al., 2017], [Künnemann et al., 2017]

CURRENT KNAPSACK ALGORITHMS ARE ESSENTIALLY OPTIMAL, BUT

Theorem: [Cygan et al., 2017], [Künnemann et al., 2017]

Min-Plus Convolution reduces to Knapsack with $s, t \in \tilde{\Theta}(n)$

Corollary: No $O((nt)^{0.99})$ and $O((st)^{0.99})$ time algorithms for Knapsack, unless Min-Plus Convolution Conjecture fails

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Dream theorem:

$\forall \alpha \leqslant \beta$, Min-Plus Convolution reduces to Knapsack with $s \in \tilde{\Theta}(n^{\alpha})$, $t \in \tilde{\Theta}(n^{\beta})$

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Theorem: [P.–Rohwedder–Węgrzycki, 2021]

Knapsack can be solved in $O(n + s^3)$ time

Knapsack

$O(nt)$	[Bellman, 1957]
$O(st)$	[Kellerer–Pferschy, 2004]
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Subset Sum	
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$\tilde{O}(t)$	[Bringmann, 2017]
$O(s^2)$	This work

Knapsack		Subset Sum	
$O(nt)$	[Bellman, 1957]	$O(nt)$	[Bellman, 1957]
$O(st)$	[Kellerer–Pferschy, 2004]	$\tilde{O}(t)$	[Bringmann, 2017]
$O(s^3)$	This work	~~$O(s^2)$~~ $\underbrace{O(s^{5/3})}_{\text{additive combinatorics magic*}}$	This work

*Unless t is too small, or (almost) all numbers have a common divisor, the answer is always YES

Knapsack

$O(nt)$ [Bellman, 1957]

$O(st)$ [Kellerer–Pferschy, 2004]

$O(s^3)$ This work

Open: Can we get $O(s^2)$?

Subset Sum

$O(nt)$ [Bellman, 1957]

$\tilde{O}(t)$ [Bringmann, 2017]

~~$O(s^2)$~~ $\underbrace{O(s^{5/3})}_{\text{additive combinatorics magic*}}$ This work

Open: Can we get $O(s)$?

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Knapsack

$O(nt)$	[Bellman, 1957]
$O(st)$	[Kellerer–Pferschy, 2004]
$O(s^3)$	This work

Open: Can we get $O(s^2)$?

Subset Sum

$O(nt)$	[Bellman, 1957]
$\tilde{O}(t)$	[Bringmann, 2017]
~~$O(s^2)$~~ $\underbrace{O(s^{5/3})}_{\text{additive combinatorics magic}^*}$	This work

Open: Can we get $O(s)$?

SEE YOU AT THE POSTER!

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